THE SINGULAR FREE BOUNDARY IN THE SIGNORINI PROBLEM FOR VARIABLE COEFFICIENTS

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Abstract. We study the singular set in the Signorini problem for a divergence form elliptic operator with Lipschitz coefficients, in the case of zero thin obstacle. The proofs are based on Weiss and Monneau type monotonicity formulas implying homogeneity, nondegeneracy, uniqueness, and continuous dependence of blowups at singular free boundary points.

Contents

1. Introduction

The aim of this paper is to study the structure and the regularity of the singular free boundary in the *Signorini problem* for a variable coefficient elliptic operator satisfying some minimal assumptions on the coefficients. Our work generalizes the previous one by the first and second named authors [GP] where the singular set was studied in the Signorini problem for the Laplacian, both for zero and non-zero thin obstacles. In this paper we restrict the analysis to the situation of a zero thin obstacle; the case of non-zero obstacle is technically more involved and will be the object of a forthcoming investigation.

This paper is also a continuation of previous work (see [GS] and [GPS]), where the optimal regularity of the solution of the Signorini problem for a variable coefficient elliptic operator and the corresponding regularity of the regular part of the free boundary were studied.

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Given a bounded open set $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, and a self-adjoint, uniformly elliptic matrix-valued function $A(x)$ in Ω, the Signorini problem consists of minimizing the (generalized) Dirichlet energy

$$\min_{v\in\mathcal{K}} \int_\Omega \langle A(x)\nabla v, \nabla v\rangle, \tag{1.1}$$

where v ranges in the closed convex set

$$\mathcal{K} = \mathcal{K}_{g,\varphi} = \{v \in W^{1,2}(\Omega) \mid v = g \ \text{ on } \ \partial\Omega \setminus \mathcal{M},\ v \geq \varphi \ \text{ on } \ \mathcal{M}\}.$$

Here, $\mathcal{M} \subset \partial\Omega$ is a codimension one manifold, g is a boundary datum and the function $\varphi : \mathcal{M} \to \mathbb{R}$ represents the lower-dimensional, or thin, obstacle. In his seminal work [F] Fichera proved that, under appropriate assumptions on the data, the minimization problem (1.1) admits a unique solution $u \in \mathcal{K}$, see also [T].

Throughout this paper the matrix-valued function $x \mapsto A(x) = [a_{ij}(x)]$ in (1.1) is assumed to be uniformly elliptic, symmetric and with Lipschitz continuous entries treated by the first and third named authors in [GS]. We note that allowing for variable coefficients is important both for applications to elasticity and for the study of a Signorini problem with a non-flat thin manifold $\mathcal{M}$ for the Laplacian. Indeed, if $\mathcal{M}$ is $C^{1,1}$, for instance, then by a standard flattening procedure one is led to analyzing a Signorini problem for a flat thin manifold where the operator has Lipschitz continuous coefficients.

Henceforth, for $x_0 \in \mathbb{R}^n$ and $r > 0$ we let $B_r(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\}$, and $S_r(x_0) = \partial B_r(x_0)$. When $x_0 = 0$, we will simply write B_r and S_r, instead of $B_r(0)$ and $S_r(0)$. When needed, points in $\mathbb{R}^n$ will be indicated with $x = (x', x_n)$, with $x' \in \mathbb{R}^{n-1}$. We also let $B_r^+ = \{(x', x_n) \in B_r \mid x_n > 0\}$, $B_r^- = \{(x', x_n) \in B_r \mid x_n < 0\}$, and indicate with $B'_r = \{(x', 0) \in \mathbb{R}^n \mid |x'| < r\}$ the thin ball centered at 0 with radius r and with $S'_r = \{(x', 0) \in \mathbb{R}^n \mid |x'| = r\}$ the thin sphere. In all integrals we will routinely omit indicating the relevant differential of n-dimensional volume dx or $(n-1)$-dimensional area $d\sigma$.

Since the problems investigated in this paper are of a local nature, for the problem (1.1) above we assume hereafter that $\Omega = B_1^+$, and that the thin manifold is flat and given by $\mathcal{M} = B'_1 \subset \partial B_1^+$. The set

$$\Lambda^\varphi(u) = \{x \in B'_1 \mid u(x) = \varphi(x)\}$$

is known as the *coincidence set*, and its boundary (in the relative topology of B'_1)

$$\Gamma^\varphi(u) = \partial_{B'_1} \Lambda^\varphi(u)$$

is known as the *free boundary*.

Before we state our main results, and in order to provide the reader with some historical perspective, we mention that in [GS] the first and third named authors established the following optimal interior regularity of the unique solution u to (1.1).

Theorem 1.1 (Optimal regularity). *Suppose that the coefficients of the matrix-valued function $A(x)$ be Lipschitz continuous. Let u be the solution of the Signorini problem* (1.1)*, with the thin obstacle $\varphi \in C^{1,1}(B'_1)$, and let $0 \in \Gamma^\varphi(u)$. Then $u \in C^{1,\frac{1}{2}}(B^+_{1/2} \cup B'_{1/2})$.*

We stress that the $C^{1,\frac{1}{2}}$ smoothness up to the thin manifold B_1' is best possible, as one can see from the example of the function $u(x) = \Re(x_1 + i|x_n|)^{3/2}$. When $L = \Delta$, such u solves the Signorini problem in B_1^+ with B_1' as thin manifold, obstacle $\varphi = 0$, and the origin as a free boundary point. Theorem 1.1 generalized to the case of Lipschitz variable coefficients the groundbreaking 2004 result of Athanasopoulos and Caffarelli for the Laplacian, zero obstacle and flat thin manifold, see [AC].

In the subsequent paper [GPS] we have established the $C^{1,\beta}_{loc}$ regularity of the regular part $\mathcal{R}^\varphi(u)$ of the free boundary. Roughly speaking, this is the collection of all free boundary points where an appropriate generalization of the Almgren frequency takes its lowest possible value $\kappa = 3/2$. The central result in [GPS] was the following.

Theorem 1.2. *Under the hypothesis of Theorem 1.1, let $x_0 \in \mathcal{R}^\varphi(u)$. Then, there exists $\eta_0 > 0$, depending on x_0, such that, after a possible rotation of coordinate axes in $\mathbb{R}^{n-1}$, one has $B'_{\eta_0}(x_0) \cap \Gamma^\varphi(u) \subset \mathcal{R}^\varphi(u)$, and*

$$B'_{\eta_0} \cap \Lambda^\varphi(u) = B'_{\eta_0} \cap \{x_{n-1} \le g(x_1, \ldots, x_{n-2})\}$$

for $g \in C^{1,\beta}(\mathbb{R}^{n-2})$ with a universal exponent $\beta \in (0,1)$.

In this paper we are interested in the structure and regularity of the so-called *singular free boundary* $\Sigma^\varphi(u)$. This set is the collection of all points $x_0 \in \Gamma^\varphi(u)$ such that the coincidence set has vanishing $(n-1)$-dimensional Hausdorff density at x_0, i.e.,

$$\lim_{r \to 0^+} \frac{\mathcal{H}^{n-1}(\Lambda^\varphi(u) \cap B_r'(x_0))}{\mathcal{H}^{n-1}(B_r'(x_0))} = 0. \tag{1.2}$$

From now on in this paper we assume that the thin obstacle $\varphi = 0$, and we simply write $\Gamma(u), \Lambda(u), \mathcal{R}(u)$ and $\Sigma(u)$.

We emphasize that the singular set $\Sigma(u)$ is by no means a small or negligible subset of the free boundary. For instance, consider in $\mathbb{R}^3$ the harmonic polynomial

$$u(x) = x_1^2 x_2^2 - \left(x_1^2 + x_2^2\right) x_3^2 + \frac{1}{3} x_3^4.$$

This function solves the Signorini problem for the Laplacian in $B_1^+ \subset \mathbb{R}^3$ with zero obstacle and flat manifold $\mathcal{M} = \{(x', 0) \mid x' \in \mathbb{R}^2\}$. On $\mathbb{R}^2 \times \{0\}$, we have $u(x_1, x_2, 0) = x_1^2 x_2^2$ and therefore the coincidence set $\Lambda(u)$ as well as the free boundary $\Gamma(u)$ consist of the union of the lines $\mathbb{R} \times \{0\} \times \{0\}$ and $\{0\} \times \mathbb{R} \times \{0\}$. Thus, all free boundary points are singular, i.e., $\Gamma(u) = \Sigma(u)$.

Hereafter in this paper we assume that the following compatibility conditions on the coefficients be satisfied

$$a_{in}(x', 0) = 0 \quad \text{in } B_1', \text{ for } i = 1, \ldots, n-1. \tag{1.3}$$

The hypothesis (1.3) essentially means that the conormal direction $A(x', 0)\nu$ coincides with the outer normal direction $\nu = -e_n$ on the flat portion B_1' of the boundary of B_1^+. As shown in the Appendix B of [GS] there is no loss of generality in assuming (1.3) since such condition can always be achieved by means of a sufficiently smooth diffeomorphism.

For the purpose of this paper it will be expedient to extend the solution u of (1.1) to the whole unit ball B_1. To accomplish this we extend the coefficients a_{ij} and the boundary datum g in the following way:

(i) $g(x', x_n) = g(x', -x_n)$;
(ii) $a_{ij}(x', x_n) = a_{ij}(x', -x_n)$ for $i, j < n$ or $i = j = n$;
(iii) $a_{in}(x', x_n) = -a_{in}(x', -x_n)$ for $i < n$.

Under these assumptions, if we extend u to the whole B_1 as an even function with respect to x_n, then the extended function (which we continue to denote by u) satisfies the condition $D_n u(x', 0) = 0$ at every point $(x', 0) \in B_1'$ where $D_n u(x', 0)$ exists. We stress that such normal derivative might not exist along the coincidence set, where $D_n^+ u$ and $D_n^- u$ might differ form each other. The function u is the unique solution to the minimization problem in B_1 similar to (1.1), and it satisfies the following conditions:

$$Lu = \operatorname{div}(A\nabla u) = 0 \quad \text{in } B_1^+ \cup B_1^-, \tag{1.4}$$

$$u \geq 0 \quad \text{in } B_1', \tag{1.5}$$

$$\langle A\nabla u, \nu_+\rangle + \langle A\nabla u, \nu_-\rangle \geq 0 \quad \text{in } B_1', \tag{1.6}$$

$$u(\langle A\nabla u, \nu_+\rangle + \langle A\nabla u, \nu_-\rangle) = 0 \quad \text{in } B_1'. \tag{1.7}$$

$$\int_{B_1} \langle A\nabla u, \nabla \eta\rangle = \int_{B_1'} (\langle A\nabla u, \nu_+\rangle + \langle A\nabla u, \nu_-\rangle)\eta, \quad \eta \in C_0^\infty(B_1). \tag{1.8}$$

The conditions (1.5)–(1.7) are known as *Signorini* or *complementarity conditions*. We note explicitly that (1.6) and (1.8) imply, in particular, that $Lu \leq 0$ in B_1, i.e., u is a supersolution of L. The nonlinear condition (1.7) is known as Signorini's ambiguous boundary condition.

We also note that, since on B_1' we have $\nu_\pm = \mp e_n$, and in view of the hypothesis (1.3) we have $A(x', 0)e_n = a_{nn}(x', 0)e_n$, then on B_1' we have

$$\langle A\nabla u, \nu_+\rangle = -a_{nn} D_n^+ u, \quad \langle A\nabla u, \nu_-\rangle = a_{nn} D_n^- u,$$

where we have respectively denoted by $D_n^+ u$ and $D_n^- u$ the vertical limits in the x_n direction from within B_1^+ and B_1^-. This convention will be followed throughout the paper. We notice that at points $(x', 0) \in B_1'$ where u is above the obstacle, i.e., where $u(x', 0) > 0$, we have from (1.7) that $-D_n^+ u + D_n^- u = 0$, and thus on such set $D_n u$ exists, and equals zero, since u is even in x_n.

Definition 1.3. In this paper we denote by $\mathfrak{S}$ the class of solutions of the normalized Signorini problem (1.4)–(1.8).

Given $u \in \mathfrak{S}$, we denote by $\Gamma_\kappa(u)$ the collection of those free boundary points $x_0 \in \Gamma(u)$ where the generalized frequency $N_{x_0}(u_{x_0}, 0^+) = \kappa$, see Definition 8.1 below.

Let $u \in \mathfrak{S}$. We say that $x_0 \in \Gamma(u)$ is a *singular point* of the free boundary if

$$\lim_{r\to 0+} \frac{\mathcal{H}^{n-1}(\Lambda(u_{x_0}) \cap B_r')}{\mathcal{H}^{n-1}(B_r')} = 0.$$

We denote with $\Sigma(u)$ the subset of singular points of $\Gamma(u)$. We also denote

$$\Sigma_\kappa(u) = \Sigma(u) \cap \Gamma_\kappa(u).$$

To state the main result of this paper we need to further classify the singular points. To this end, we rely on the study of blowups of our solution and we define the dimension of $\Sigma_\kappa(u)$ at a singular point $x_0 \in \Sigma_\kappa(u)$ to be (see Definition 11.1)

$$d_\kappa^{x_0} = \dim\left\{\xi \in \mathbb{R}^n \mid \langle \xi, \nabla_{x'} p_\kappa^{x_0}(x',0)\rangle = 0,\ \forall x' \in \mathbb{R}^{n-1}\right\},$$

where $p_\kappa^{x_0}$ is a nonzero homogeneous polynomial of degree κ which is the homogeneous blowup of u at x_0 (see Theorem 10.5). We then define

$$\Sigma_\kappa^d(u) := \{x_0 \in \Sigma_\kappa(u) \mid d_\kappa^{x_0} = d\}.$$

The following is the main result in this paper.

Theorem 1.4 (Structure of the singular set)**.** *Let $u \in \mathfrak{S}$. Then $\Gamma_\kappa(u) = \Sigma_\kappa(u)$ for $k = 2m$, $m \in \mathbb{N}$. Moreover, every set $\Sigma_\kappa^d(u)$, $d \in \{0,\dots,n-2\}$, is contained in a countable union of d-dimensional C^1 manifolds.*

The proof of Theorem 1.4 rests on several results and will ultimately be presented at the end of the paper in Section 10. Loosely speaking, such proof follows the outline of the analogous result in [GP] for the case of the Laplacian, but the treatment of Lipschitz variable coefficients poses novel interesting challenges. One of the main results proved here is the "almost monotonicity" of a one-parameter family of Weiss type functionals, see Theorem 5.3 below. One important consequence of such result is the fact that the homogeneous blowup (see Definition 6.10) of a function $u \in \mathfrak{S}$ is a global solution of the Signorini problem which is homogeneous of degree $\kappa = N(u, 0^+)$, see Proposition 6.12. Another central tool in the proof of Theorem 1.4 is Theorem 9.3, a monotonicity formula for a one-parameter family of Monneau type functionals. Such monotonicity formula leads to the crucial non-degeneracy Lemma 10.2 which, in turn, implies the uniqueness of homogeneous blowups of $u \in \mathfrak{S}$ and that such homogeneous blowup does not vanish identically. From that point on, the proof of Theorem 1.4 follows along the lines of its predecessor in [GP] for the Laplacian. In the case at hand some delicate uniformity matters still need to be dealt with, which is done in Section 7.

1.1. **Structure of the paper.** The paper is organized as follows. In Section 2 we introduce some preliminary material used throughout the paper. The monotonicity of an Almgren type frequency function, $N(u, r)$, is established in Section 3. Section 4 is devoted to some growth lemmas. In Section 5 we prove one of the main results of this paper, a Weiss-type monotonicity formula. In Section 6 we introduce the Almgren and homogeneous blowups of our solution. Furthermore, we use the Almgren and Weiss-type monotonicity formulas to conclude that if $\lim_{r\to 0+} N(r) = \kappa$, both the Almgren and the homogeneous blowups of our solution are homogeneous of degree κ. In Section 7 we deal with the uniformity matters mentioned above stemming from the fact that we have a variable coefficient matrix. In Section 8 we formally define the singular set and prove a characterization of singular points. The second main technical result of this paper, a Monneau-type monotonicity formula, is proved in Section 9. Finally, in Section 10 we use the Almgren and Monneau monotonicity formulas to derive a nondegeneracy property of our solution, which finally allows us to prove the uniqueness of homogeneous blowups, and that such blowup cannot vanish identically. We further prove the the continuos dependence of the blowups, which allows us to conclude the proof of the structure of the singular set, which is done in Section 11.

2. Preliminary material

Given a matrix-valued function $A(x) = [a_{ij}(x)]$ in B_1, we consider the problem of minimizing the generalized energy

$$\min_{u\in\mathcal{K}} \int_{B_1} \langle A(x)\nabla u, \nabla u\rangle, \tag{2.1}$$

where u ranges in the closed convex set

$$\mathcal{K} = \{u \in W^{1,2}(B_1) \mid u = g \ \text{ on } \ S_1,\ u \ge \varphi \text{ on } B_1'\}.$$

Our assumptions on the matrix-valued function $x \to A(x) = [a_{ij}(x)]$ in (2.1) are as follows:

(i) $a_{ij}(x) = a_{ji}(x)$ for $i, j = 1, \dots, n$, and every $x \in B_1$;

(ii) there exists $\lambda > 0$ such that for every $x \in B_1$ and $\xi \in \mathbb{R}^n$, one has

$$\lambda|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \lambda^{-1}|\xi|^2; \tag{2.2}$$

(iii) the entries of $A(x) = [a_{ij}(x)]$ are in $W^{1,\infty}(B_1)$, i.e., one has for some $Q > 0$ and every $x, y \in B_1$

$$\|A(x) - A(y)\| \le Q|x - y|. \tag{2.3}$$

The next lemma expresses a simple, yet important fact.

Lemma 2.1. *Suppose $A(0) = I_n$. Then, for $x \ne 0$, one has*

$$L|x| = \operatorname{div}(A(x)\nabla|x|) = \frac{n-1}{|x|} + O(1)\,. \tag{2.4}$$

In particular, $L|x| \in L^1_{loc}(\mathbb{R}^n)$.

Proof. With $r(x) = |x|$ and $B(x) = A(x) - A(0)$, we have

$$\operatorname{div}(A(x)\nabla r) = \Delta r + \operatorname{div}(B(x)\nabla r) = \frac{n-1}{r} + \operatorname{div}(B(x)\nabla r).$$

Now, if $B(x) = [b_{ij}(x)]$, we have

$$\operatorname{div}(B(x)\nabla r) = D_i(b_{ij})D_j r + b_{ij}D_{ij}r.$$

From (2.3) and Rademacher's theorem we have

$$D_i(b_{ij})D_j r = O(1)\,, \quad b_{ij}D_{ij}r = \frac{b_{ij}}{r}\left(\delta_{ij} - \frac{x_i x_j}{r^2}\right) = O(1)\,.$$

The desired conclusion thus follows. □

We next introduce the conformal factor

$$\mu(x) = \langle A(x)\nabla r(x), \nabla r(x)\rangle = \frac{\langle A(x)x, x\rangle}{|x|^2}. \tag{2.5}$$

Let us observe explicitly that when $A \equiv I_n$ we have $\mu \equiv 1$. From the assumption (2.2) on A one easily checks that

$$\lambda \le \mu(x) \le \lambda^{-1}, \quad x \in B_1. \tag{2.6}$$

We have the following simple lemma whose proof we omit since it is similar to that of Lemma 2.1.

Lemma 2.2. *Suppose that* $A(0) = I_n$. *Then, one has*

(1) $\mu(0) = 1$,
(2) $|1 - \mu(x)| \le C|x|$,
(3) $|\nabla \mu| \le C$,

where $C > 0$ *is universal.*

We now introduce a vector field which plays a special role in what follows. With μ as in (2.5) we define

$$Z(x) = r(x)\frac{A(x)\nabla r}{\mu(x)} = \frac{A(x)x}{\mu(x)}. \tag{2.7}$$

A crucial property of Z is that, denoting by ν the outer unit normal to the sphere S_r, we have

$$\langle Z, \nu\rangle = r\frac{\langle A(x)\nabla r, \nabla r\rangle}{\mu} \equiv r, \quad \text{on } S_r. \tag{2.8}$$

Another important fact concerning the vector field Z is contained in the following

Lemma 2.3. *Suppose that* $A(0) = I_n$. *There exists a universal* $O(r)$ *such that for every* $i, j = 1, \dots, n$, *one has*

$$D_i Z_j = \delta_{ij} + O(r). \tag{2.9}$$

In particular, one has

$$\operatorname{div} Z = n + O(r). \tag{2.10}$$

Proof. From (2.3), (2.7), and from 2) and 3) of Lemma 2.2 we have for a universal $O(r)$

$$\begin{aligned} D_i Z_j &= D_i\left(\frac{a_{jk}x_k}{\mu}\right) = \frac{(D_i a_{jk})x_k}{\mu} + \frac{a_{jk}\delta_{ki}}{\mu} - \frac{a_{jk}x_k D_i\mu}{\mu^2} \\ &= \frac{a_{ij}}{\mu} + O(r) = \frac{\delta_{ij}}{\mu} + O(r) = \delta_{ij} + \delta_{ij}\left(\frac{1}{\mu} - 1\right) + O(r) = \delta_{ij} + O(r). \end{aligned}$$

□

3. Monotonicity of the frequency

The principal objective of this section is to establish the monotonicity Theorem 3.4 below. We consider the variational solution u of (2.1) and assume that $0 \in \Gamma(u)$.

Definition 3.1. For $r > 0$ we define the *height* of u in the ball B_r as

$$H(u, r) = \int_{S_r} u^2 \mu. \tag{3.1}$$

The Dirichlet integral of u in B_r is defined by

$$D(u, r) = \int_{B_r} \langle A\nabla u, \nabla u\rangle. \tag{3.2}$$

The *frequency* of u in B_r is given by

$$N(u, r) = \frac{rD(u, r)}{H(u, r)}. \tag{3.3}$$

Henceforth, when the function u is fixed, we will write $H(r)$, $D(r)$ and $N(r)$, instead of $H(u,r), D(u,r)$ and $N(u,r)$. Before proceeding we make the observation (important for the computations in this section) that, thanks to the results in [AU1, AU3], under the assumptions of Theorem 1.2 above, we know that the weak solution u of (1.4)–(1.8) is in $u \in C^{1,\alpha}_{\text{loc}}(B_1^{\pm} \cup B_1')$. Consequently, all derivatives are classical in the ensuing computations.

Let u be a solution of the thin obstacle problem (1.4)–(1.8) in B_1 with $\varphi \equiv 0$. In what follows we recall some important results from [GS], adapted to the case of zero obstacle. We have

$$D(r) = \int_{S_r} u\langle A\nabla u, \nu\rangle. \tag{3.4}$$

Next, we recall the following first variation formula for the height:

$$H'(r) = 2\int_{S_r} u\langle A\nabla u, \nabla r\rangle + \int_{S_r} u^2 L|x|.$$

From this formula and from (2.4) in Lemma 2.1 above, we immediately obtain the following result.

Proposition 3.2. *Assume that the normalization hypothesis*

$$A(0) = I_n \tag{3.5}$$

be in place. Under this assumption for a.e. $0 < r < 1$ *one has*

$$H'(r) - \frac{n-1}{r}H(r) - 2\int_{S_r} u\langle A\nabla u, \nu\rangle = O\,(1)\,H(r). \tag{3.6}$$

Next, we establish a basic first variation formula for the energy.

Theorem 3.3. *Suppose that the matrix* $A(x)$ *satisfy the hypothesis* (3.5) *and that furthermore* (1.3) *be in force. Then, for a.e.* $r \in (0,1)$ *one has*

$$D'(r) = 2\int_{S_r} \frac{\langle A\nabla u, \nu\rangle^2}{\mu} + \left(\frac{n-2}{r} + O\,(1)\right) D(r). \tag{3.7}$$

Proof. From Theorem A.10 in [GS] we have

$$\begin{aligned} rD'(r) = &\int_{B_r} (\operatorname{div} Z)\langle A\nabla u, \nabla u\rangle + \int_{B_r} (Za_{ij})D_iuD_ju + 2\int_{S_r} Zu\langle A\nabla u, \nu\rangle \\ &- 2\int_{S_r} a_{ij}(D_iZ_k)D_juD_ku + \int_{B_r'} \frac{a_{nn}}{\mu}(a_{ni}x_i)((D_n^- u)^2 - (D_n^+ u)^2). \end{aligned}$$

Thanks to the assumption (1.3) the integral on the thin ball B_r' vanishes, and we obtain from this identity

$$\begin{aligned} rD'(r) = &\int_{B_r} (\operatorname{div} Z)\langle A\nabla u, \nabla u\rangle + \int_{B_r} (Za_{ij})D_iuD_ju + 2\int_{S_r} Zu\langle A\nabla u, \nu\rangle \\ &- 2\int_{S_r} a_{ij}(D_iZ_k)D_juD_ku. \end{aligned} \tag{3.8}$$

Using (2.10) in Lemma 2.3 we find

$$\int_{B_r} (\operatorname{div} Z)\langle A\nabla u, \nabla u\rangle = nD(r) + O(r)D(r).$$

From (2.9) and (2.3) we obtain

$$-2\int_{S_r} a_{ij}(D_i Z_k) D_j u D_k u = -2\int_{S_r} a_{ij}\left(\delta_{ik}+O(r)\right) D_j u D_k u = -2D(r) + O(r)D(r).$$

Finally, the definition (2.7) and (2.3) give

$$Za_{ij} = \frac{\langle A(x)x, \nabla(a_{ij})\rangle}{\mu(x)} = O(r).$$

From this observation and the hypothesis (2.3) we thus have

$$\int_{B_r} (Za_{ij}) D_i u D_j u = O(r)D(r).$$

In conclusion, we obtain from (3.8)

$$rD'(r) = (n-2)D(r) + O(r)D(r) + 2\int_{S_r} Zu\langle A\nabla u, \nu\rangle.$$

Recalling that $Zu = \frac{\langle A(x)\nabla u, \nabla r\rangle}{\mu} r$, this gives the desired conclusion. □

With these results in hands we can now prove the main result of this section.

Theorem 3.4 (Monotonicity of the adjusted frequency)**.** *Assume that the hypothesis of Theorem 3.3 be satisfied. Then there exists a universal constant $C > 0$ such that the function*

$$\tilde{N}(r) =: e^{Cr} N(r) \tag{3.9}$$

is monotone nondecreasing in $(0,1)$. In particular, the limit $\lim_{r\to 0} \tilde{N}(r) = \tilde{N}(0^+)$ exists. We conclude that $\lim_{r\to 0} N(r) = N(0^+)$ also exists, and equals $\tilde{N}(0^+)$.

Proof. From (3.3), Proposition 3.2 and Theorem 3.3 we have for a.e. $r \in (0,1)$

$$\begin{aligned}
\frac{d}{dr}\log N(r) &= \frac{D'(r)}{D(r)} + \frac{1}{r} - \frac{H'(r)}{H(r)} \\
&= 2\frac{\int_{S_r} \frac{\langle A\nabla u,\nu\rangle^2}{\mu}}{D(r)} + \frac{n-2}{r} + O(1) + \frac{1}{r} \\
&- \frac{n-1}{r} - 2\frac{\int_{S_r} u\langle A\nabla u, \nu\rangle}{H(r)} + O(1) \\
&= 2\frac{\int_{S_r} \frac{\langle A\nabla u,\nu\rangle^2}{\mu}}{D(r)} - 2\frac{\int_{S_r} u\langle A\nabla u, \nu\rangle}{H(r)} + O(1) \\
&\geq -C,
\end{aligned}$$

where in the last inequality $C > 0$ is a universal constant, and we have used the Cauchy-Schwarz inequality and the identity (3.4) above. The latter inequality finally gives

$$\frac{d}{dr}\log \tilde{N}(r) = \frac{d}{dr}\log N(r) + C \geq 0,$$

which implies the desired conclusion. □

Lemma 3.5 (Minimal homogeneity)**.** *Assume that the hypothesis of Theorem 3.3 be satisfied. Then*

$$N(0^+) = \frac{3}{2} \quad \text{or} \quad N(0^+) \geq 2.$$

Proof. With μ as in (2.5), we define

$$d_r = \left(\frac{H(r)}{r^{n-1}}\right)^{1/2} = \left(\frac{1}{r^{n-1}}\int_{S_r} u^2\mu\right)^{1/2}, \quad 0<r<1,$$

and, similarly to what was done in [ACS] in the study of the thin obstacle problem for the Laplacian, we consider the *non-homogeneous scalings* of u,

$$u_r(x) = \frac{u(rx)}{d_r}, \quad x\in B_{\frac{1}{r}}. \tag{3.10}$$

With $A_r(x)=[a^r_{ij}(x)]$, where $a^r_{ij}(x)=a_{ij}(rx)$, $0<r<1$, we have

$$\int_{S_1} u_r^2(x)\langle A_r(x)\nu,\nu\rangle d\sigma(x) = \frac{r^{1-n}}{d_r^2}\int_{S_r} u^2(y)\langle A(y)\nu,\nu\rangle d\sigma(y) = \frac{r^{1-n}}{d_r^2}H(r)=1. \tag{3.11}$$

Moreover, u_r is the (unique) minimizer of the Dirichlet integral

$$J_r(v)=\int_{B_{1/r}}\langle A_r(x)\nabla v,\nabla v\rangle dx,$$

over the closed convex set

$$\mathcal{K}_r=\{v\in W^{1,2}(B_{1/r}) \mid v=\psi_r \text{ on } S_{1/r}, u\ge 0 \text{ on } B'_{1/r}\},$$

where $\psi_r(x)=\frac{\psi(rx)}{d_r}$. Given a number $0<\rho<1/r$, the appropriate (generalized) frequency function for u_r in B_ρ is

$$N_{L_r}(u_r,\rho)=\rho\frac{\int_{B_\rho}\langle A_r\nabla u_r,\nabla u_r\rangle}{\int_{S_\rho}u_r^2\langle A_r\nu,\nu\rangle}, \quad 0<\rho<\frac{1}{r}. \tag{3.12}$$

We emphasize that the notation $N_{L_r}(u_r,\cdot)$ is now necessary in order to emphasize the fact that the rescaled function u_r in (3.10) is associated with the operator $L_r=\operatorname{div}(A_r\nabla)$. With this notation the function $N(u,r)$, defined in (3.3), will be denoted by $N_L(u,r)$. A crucial (and easy to see) property of the frequency is the following scale invariance: for every $0<\rho<\frac{1}{r}$, one has $N_{L_r}(u_r,\rho)=N_L(u,r\rho)$.

From (3.11), the scale invariance remarked above and the monotonicity formula in Theorem 3.4, we obtain

$$\begin{aligned}\lambda\int_{B_1}|\nabla u_r|^2 \le \int_{B_1}\langle A_r\nabla u_r,\nabla u_r\rangle &= N_{L_r}(u_r,1)\\ &= N_L(u,r)\le e^C N_L(u,1),\end{aligned} \tag{3.13}$$

where $C>0$ is the universal constant in Theorem 3.4. On the other hand, (3.11) above gives

$$\lambda\int_{S_1}u_r^2 \le \int_{S_1}u_r^2\langle A_r\nu,\nu\rangle = 1. \tag{3.14}$$

From (3.13), (3.14), and the trace inequality, we see that

$$\|u_r\|_{W^{1,2}(B_1)}\le C^\star, \quad 0<r<1,$$

where $C^\star>0$ depends only on n,λ and $N_L(u,1)$. Hence, there exists a function $u_0\in W^{1,2}(B_1)$ such that for some subsequence $r_j\to 0^+$,

$$u_{r_j}\to u_0 \quad \text{weakly in } W^{1,2}(B_1). \tag{3.15}$$

Since the embedding $W^{1,2}(B_1, dx) \hookrightarrow L^2(S_1, d\sigma)$ is compact, we have the strong convergence

$$u_{r_j} \to u_0 \quad \text{in } L^2(S_1, d\sigma), \tag{3.16}$$

and also $u_r \to u_0$ in $C^1_{\text{loc}}(B_1^\pm \cup B_1')$. We call the function u_0 a *blow-up* of the solution u at the free-boundary point $0 \in \Gamma(u)$. Moreover, by (3.11) and (3.16) we have

$$1 = \int_{S_1} u_{r_j}^2 \langle A_{r_j} x, x\rangle \underset{j\to\infty}{\longrightarrow} \int_{S_1} u_0^2, \tag{3.17}$$

and we infer that $u_0 \not\equiv 0$ on S_1. It is easy to see that u_0 satisfies the following

$$\Delta u_0 = 0 \text{ in } \; B_1^+ \cup B_1^-,$$

$$u_0 \geq 0, \quad \partial_{\nu_+} u_0 + \partial_{\nu_-} u_0 \geq 0, \quad u_0(\partial_{\nu_+} u_0 + \partial_{\nu_-} u_0) = 0 \quad \text{on } B_1'.$$

Therefore u_0 is a normalized solution to the Signorini problem for Δ in B_1. In particular, by the results in [AC] we infer that $u_0 \in C^{1,\frac{1}{2}}(B_1^\pm \cup B_1')$.

We now claim that if $\int_{S_r} u_0^2 = 0$ for some $0 < r < 1$, then $u_0 \equiv 0$ in B_1. Indeed, one can easily show that $\int_{B_r} |\nabla u_0|^2 = \int_{S_r} u_0 \langle \nabla u_0, \nu\rangle = 0$, so $u_0 \equiv c$ in B_r. Since $u_0 = 0$ on S_r, then $c = 0$. By the unique continuation property of harmonic functions, we would have that $u_0 \equiv 0$ in $B_1^\pm$, hence in B_1, which contradicts (3.17). Since for every $0 < r < 1$ one has $\int_{S_r} u_0^2 > 0$, and $u_{r_j} \to u_0$ in $C^1_{\text{loc}}(B_1^\pm \cup B_1')$, we conclude that

$$N_\Delta(u_0, r) = r \frac{\int_{B_r} |\nabla u_0|^2}{\int_{S_r} u_0^2} = \lim_{r_j \to 0^+} N_{L_{r_j}}(u_{r_j}, r) = \lim_{r_j\to 0^+} N_L(u, r r_j) = N_L(u, 0+) := \kappa. \tag{3.18}$$

Equation (3.18) shows that the standard Almgren's frequency function $N_\Delta(u_0, \cdot)$ of u_0 is constant in $(0,1)$, and equals κ. By Theorem 1.4.1 in [GP] we conclude that u_0 is homogeneous of degree κ in B_1.

Finally, again from (3.18) we conclude that

$$\kappa = N_L(u, 0^+) = \lim_{j\to\infty} r_j \frac{\int_{B_{r_j}} |\nabla u_0|^2}{\int_{S_{r_j}} u_0^2} = \lim_{j\to\infty} N_\Delta(u_0, r_j) = N_\Delta(u_0, 0^+).$$

Therefore, if u_0 is a blowup of u at the origin as above, then u_0 is a normalized solution of the Signorini problem for Δ in B_1 homogeneous of degree $\kappa = N_\Delta(u_0, 0^+)$. We can thus appeal to Proposition 9.9 and Corollary 9.10 of [PSU] to reach the desired conclusion. □

4. Some growth lemmas

We begin this section by establishing a first basic consequence of Theorem 3.4.

Lemma 4.1. *Assume that the hypothesis of Theorem 3.3 be satisfied, and suppose that $N(0^+) \geq \kappa$. Then, for $r \in (0,1)$ one has*

$$H(r) \leq \tilde{C} r^{n-1+2\kappa}, \tag{4.1}$$

where $\tilde{C} = e^C H(1)$, with C as in (3.9) above.

Proof. We return to the equation (3.6) above which, using (3.4), can be written

$$\frac{d}{dr}\log\frac{H(r)}{r^{n-1}} = 2\frac{N(r)}{r} + O\left(1\right), \tag{4.2}$$

for a.e. $0 < r < 1$. Since $\tilde{N}$ is monotone on $(0,1)$, by the hypothesis and Theorem 3.4 we have

$$\kappa \le N(0^+) = \tilde{N}(0^+) \le \tilde{N}(r) \tag{4.3}$$

for every $r \in (0,1)$. We now fix $r \in (0,1)$ and integrate (4.2) between r and 1, obtaining

$$\begin{aligned}\log H(1) - \log\frac{H(r)}{r^{n-1}} &\ge 2\int_r^1 e^{-Ct}\tilde{N}(t)\frac{dt}{t} - C \\ &\ge 2e^{-C}\int_r^1 \tilde{N}(t)\frac{dt}{t} - C \ge e^{-C}\log\left(\frac{1}{r}\right)^{2\kappa} - C,\end{aligned}$$

where we have used (4.3). This gives

$$\log\frac{H(r)}{r^{n-1}} \le \log H(1) + \log r^{2\kappa} + C = \log e^C H(1) r^{2\kappa}.$$

Exponentiating, we obtain the desired conclusion. □

Next we prove a growth estimate for u that will play an important role in the rest of the paper.

Lemma 4.2. *Under the hypothesis of Theorem 3.3, suppose that $N(0^+) \ge \kappa$. Then, there exists a universal constant $\overline{C} > 0$, depending also on κ, such that for every $x \in B_{1/2}$ one has*

$$|u(x)| \le \overline{C}\ |x|^\kappa. \tag{4.4}$$

Proof. We begin by observing that integrating (4.1) in Lemma 4.1 and using (2.6) above, we obtain for $0 < r < 1$

$$\int_{B_r} u^2 \le C_1 r^{n+2\kappa}, \tag{4.5}$$

where C_1 depends on $\tilde{C}, n$ and κ. Since $Lu^\pm \ge 0$ in B_1, we can apply Theorem 8.17 of [GT] to infer the existence of $c = c(n,\lambda) > 0$ such that if $B(x,2R) \subset B_1$, then

$$\sup_{B(x,R)} u^+ \le c\ R^{-n/2}\|u^+\|_{L^2(B(x,2R))}. \tag{4.6}$$

Pick now $x \in B_{1/2}$, and let $R = |x|/2$. Clearly, $B(x,2R) \subset B_{4R} \subset B_1$. Applying (4.6) we find

$$\begin{aligned}u^+(x) &\le c\ R^{-\frac{n}{2}}\left(\int_{B(x,2R)}(u^+)^2\right)^{\frac{1}{2}} \le c\ R^{-\frac{n}{2}}\left(\int_{B_{4R}}(u^+)^2\right)^{\frac{1}{2}} \le c\ R^{-\frac{n}{2}}\left(\int_{B_{4R}}u^2\right)^{\frac{1}{2}} \\ &\le \overline{C}R^{-\frac{n}{2}}R^{\frac{n+2\kappa}{2}} = \overline{C}|x|^\kappa,\end{aligned}$$

where in the second to the last inequality we have used (4.5) above. Since a similar result holds for u^-, we have reached the desired conclusion. □

Lemma 4.3. *Assume that the hypothesis of Theorem 3.3 be satisfied, and suppose that $N(0^+) \ge \kappa$. Then, there exists a universal constant $C^\star > 0$ such that for $r \in (0,1)$ one has*

$$D(r) \le C^\star r^{n-2+2\kappa}. \tag{4.7}$$

Proof. As a first observation we recall the estimate (4.5) above. The desired conclusion (4.7) would thus follow at once from this observation, provided that the following Caccioppoli type inequality hold

$$D\left(\frac{r}{2}\right) \le \frac{C_2}{r^2}\int_{B_r} u^2, \tag{4.8}$$

for every $0 < r < 1$ for a universal constant $C_2 > 0$. To prove (4.8), let $\alpha \in C_0^\infty(B_r)$ be such that $0 \le \alpha \le 1$, $\alpha \equiv 1$ on $B_{\frac{r}{2}}$ and $|\nabla\alpha| \le \frac{C}{r}$, and define $h = \alpha^2 u$. By the Signorini conditions (1.7) and (1.8) above, we have

$$\int_{B_r} \langle A\nabla u, \nabla h\rangle = \int_{B_r'} \alpha^2 u(\langle A\nabla u, \nu_+\rangle + \langle A\nabla u, \nu_-\rangle) dH^{n-1} = 0.$$

This gives

$$0 = \int_{B_r} \alpha^2\langle A\nabla u, \nabla u\rangle + \int_{B_r} 2\alpha u\langle A\nabla u, \nabla\alpha\rangle,$$

which implies

$$\int_{B_r} \alpha^2\langle A\nabla u, \nabla u\rangle \le 2\left(\int_{B_r} u^2\langle A\nabla\alpha, \nabla\alpha\rangle\right)^{\frac{1}{2}}\left(\int_{B_r} \alpha^2\langle A\nabla u, \nabla u\rangle\right)^{\frac{1}{2}}.$$

In a standard fashion this gives (4.8). □

5. A one-parameter family of Weiss type monotonicity formulas

In this section we introduce a generalization of the Weiss type functional in [GP] and establish a basic almost monotonicity property of the latter.

Definition 5.1. For $\kappa > 0$ we define

$$\begin{aligned} W_\kappa(r) = W_\kappa(r,u) &= \frac{1}{r^{n-2+2\kappa}}\int_{B_r}\langle A\nabla u, \nabla u\rangle - \frac{\kappa}{r^{n-1+2\kappa}}\int_{S_r} u^2\mu \\ &= \frac{1}{r^{n-2+2\kappa}}D(r) - \frac{\kappa}{r^{n-1+2\kappa}}H(r) = \frac{H(r)}{r^{n-1+2\kappa}}\left\{N(r) - \kappa\right\}. \end{aligned}$$

Lemma 5.2. *Suppose that the hypothesis of Theorem 3.3 be satisfied. If $N(0^+) \ge \kappa$, then there exists $\overline{C} > 0$ such that $|W_\kappa(r)| \le \overline{C}$ for every $0 < r < 1$. If instead $N(0^+) = \kappa$, then $W_\kappa(0^+) = \lim_{r\to 0^+} W_\kappa(r)$ exists and equals 0.*

Proof. The former conclusion is a direct consequence of (4.1) in Lemma 4.1 and (4.7) in Lemma 4.3. The latter follows immediately from the expression $W_\kappa(r) = \frac{H(r)}{r^{n-1+2\kappa}}\{N(r) - \kappa\}$, and from the fact that the quotient $\frac{H(r)}{r^{n-1+2\kappa}}$ is bounded in view of (4.1) in Lemma 4.1. □

The following "almost monotonicity" property of the functional W_κ plays a crucial role in our further study.

Theorem 5.3. *Assume that the hypothesis of Theorem 3.3 be satisfied, and suppose that $N(0^+) \ge \kappa$. Then, there exists a universal constant $C > 0$ such that*

$$\frac{d}{dr}\left(W_\kappa(r) + Cr\right) \ge \frac{2}{r^{n+2\kappa}}\int_{S_r}\left(\frac{\langle A\nabla u, x\rangle}{\sqrt{\mu}} - \kappa\sqrt{\mu}u\right)^2. \tag{5.1}$$

As a consequence of (5.1) the function $r \to W_\kappa(r) + Cr$ is monotone non-decreasing, and therefore it has a limit as $r \to 0^+$. As a consequence, also the limit $W_\kappa(0^+) = \lim_{r\to 0^+} W_\kappa(r)$ exists and is finite.

Proof. From Definition 5.1 we have

$$W_\kappa'(r) = \frac{1}{r^{n-2+2\kappa}}\left(D'(r) - \frac{n-2+2\kappa}{r}D(r) - \frac{\kappa}{r}H'(r) + \frac{\kappa(n-1+2\kappa)}{r^2}H(r)\right)$$

Using (3.6) in Proposition 3.2 and (3.7) in Theorem 3.3 we easily obtain

$$W_\kappa'(r) = \frac{2}{r^{n-2+2\kappa}}\left\{\int_{S_r}\left(\frac{\langle A\nabla u,\nu\rangle}{\sqrt{\mu}} - \frac{\kappa}{r}u\sqrt{\mu}\right)^2 + O(1)\,D(r) + O(1)\,\frac{H(r)}{r}\right\}.$$

In view of (4.1) in Lemma 4.1 and (4.7) in Lemma 4.3, we conclude that

$$W_\kappa'(r) \geq \frac{2}{r^{n+2\kappa}}\int_{S_r}\left(\frac{\langle A\nabla u,x\rangle}{\sqrt{\mu}} - \kappa\sqrt{\mu}u\right)^2 - C,$$

for some universal constant $C > 0$. This estimate implies the sought for conclusion (5.1). In particular, this implies that the function $r \to W_\kappa(r) + Cr$ has a limit as $r \to 0^+$. We conclude that $W_\kappa(0^+)$ exists and is finite.

□

6. Almgren blowups and homogeneous blowups

In this section we consider a solution $u \in \mathfrak{S}$ and, assuming that $0 \in \Gamma(u)$, we introduce two families of scalings of u at zero, the *Almgren scalings* and the *homogeneous scalings*. We further suppose that the hypothesis (1.3) be in force and that the matrix-valued function $x \to A(x)$ satisfy the assumption (3.5) at zero. Then, we use Theorems 3.4 and 5.3 to establish the existence of appropriate blowups of u associated with each of these two families of scalings. We begin by defining the following quantity

$$d_r = \left(\frac{H(r)}{r^{n-1}}\right)^{1/2}, \tag{6.1}$$

where $H(r)$ is as in (3.1) above. We notice that (4.1) of Lemma 4.1 above implies:

$$N(0^+) \geq \kappa \implies d_r = O(r^\kappa).$$

Definition 6.1. We define the *Almgren scalings* of u as follows:

$$\tilde{u}_r(x) = \frac{u(rx)}{d_r}, \quad x \in B_{1/r}. \tag{6.2}$$

The *homogeneous scalings* of u are defined in the following way:

$$u_r(x) = \frac{u(rx)}{r^\kappa}, \quad x \in B_{1/r}. \tag{6.3}$$

In what follows we introduce the notation $\mu_r(x) = \mu(rx)$, where μ is the conformal factor defined in (2.5) above. We note explicitly that from (6.1) and (6.2) we obtain

$$H(r) = \int_{S_r} u^2\mu = r^{n-1}\int_{S_1} u(rx)^2\mu(rx) = r^{n-1}d_r^2\int_{S_1}\tilde{u}_r^2\mu_r = H(r)\int_{S_1}\tilde{u}_r^2\mu_r.$$

This implies in particular that for every $0 < r < 1$ one has

$$\int_{S_1} \tilde{u}_r^2 \mu_r = 1. \tag{6.4}$$

This normalization is the main reason for introducing (6.2). We also observe in passing that (6.3) gives trivially

$$\nabla u(rx) = r^{\kappa-1} \nabla u_r(x).$$

Lemma 6.2. *Let $u \in \mathfrak{S}$ and define $A_r(x) = A(rx)$. Then, both the functions $\tilde{u}_r$ and u_r defined in* (6.2) *and* (6.3) *are even in x_n and solve the thin obstacle problem* (1.4)–(1.8) *in $B_{1/r}$ for the operator $L_r = \operatorname{div}(A_r \nabla)$.*

Proof. It is easy to verify that u_r verifies (1.4)–(1.7) for the operator L_r. It thus suffices to prove (1.8). Given $\eta \in C_0^\infty(B_{1/r})$, a change of variable easily leads to

$$\int_{B_{1/r}} \langle A_r \nabla \tilde{u}_r, \nabla \eta \rangle = -2 \int_{B'_{1/r}} (a_{nn})_r \eta D_n^+ \tilde{u}_r, \tag{6.5}$$

and a similar equation holds if we replace $\tilde{u}_r$ with u_r. This establishes the lemma. □

Remark 6.3. Notice that when considering $\tilde{u}_r$ or u_r it is important to keep in mind that the operator being considered is $L_r = \operatorname{div}(A_r \nabla)$. Therefore, to avoid confusion, the functions $H(r)$, $D(r)$, $N(r)$, $\tilde{N}(r)$ and $W_\kappa(r)$ will be denoted by $H_{L_r}(r), D_{L_r}(r), N_{L_r}(r), \tilde{N}_{L_r}(r)$ and $W_{L_r,\kappa}(r)$. If no operator is indicated, it is understood to be L.

We now want to analyze the asymptotic behavior of the Almgren scalings $\tilde{u}_r$.

Lemma 6.4. *Let $u \in \mathfrak{S}$ and suppose that $0 \in \Gamma(u)$. Then,*

$$N_{L_r}(\tilde{u}_r, 1) = N_L(u, r).$$

Proof. The result follows from the following direct computation:

$$\begin{aligned} N_{L_r}(\tilde{u}_r, 1) &= \frac{\int_{B_1} \langle A_r \nabla \tilde{u}_r, \nabla \tilde{u}_r \rangle}{\int_{S_1} \tilde{u}_r^2 \mu_r} = \frac{r^2 \int_{B_1} \langle A(rx) \nabla u(rx), \nabla u(rx) \rangle}{\int_{S_1} u^2(rx) \mu(rx)} = \frac{r \int_{B_r} \langle A \nabla u, \nabla u \rangle}{\int_{S_r} u^2 \mu} \\ &= N_L(u, r). \end{aligned}$$ □

The next lemma combines Theorem 3.4 with Lemma 6.4 to obtain a uniform bound of the Almgren scalings in $W^{1,2}$ norm.

Lemma 6.5. *Let $u \in \mathfrak{S}$ and $0 \in \Gamma(u)$. Assume that the hypothesis of Theorem 3.3 be satisfied. Given $r_j \to 0$, the sequence $\{\tilde{u}_{r_j}\}_{j \in \mathbb{N}}$ is uniformly bounded in $W^{1,2}(B_1)$.*

Proof. By (6.4) and Lemma 6.4 we have with $r = r_j$

$$\begin{aligned} \int_{B_1} |\nabla \tilde{u}_r|^2 &\leq \lambda^{-1} \int_{B_1} \langle A_r \nabla \tilde{u}_r, \nabla \tilde{u}_r \rangle = \lambda^{-1} D_{L_r}(\tilde{u}_r, 1) = \lambda^{-1} N_{L_r}(\tilde{u}_r, 1) = \lambda^{-1} N_L(u, r) \\ &= \lambda^{-1} e^{-Cr} \tilde{N}_L(u, r) \leq \lambda^{-1} \tilde{N}_L(u, 1), \end{aligned}$$

where in the last inequality we have used the monotonicity of $\tilde{N}_L(u, \cdot)$. Moreover, by (2.5) and (6.4) again, we have

$$\int_{S_1} \tilde{u}_r^2 \leq \lambda^{-1} \int_{S_1} \tilde{u}_r^2 \mu_r = \lambda^{-1}.$$

Combining these estimates with the trace inequality, valid for any function $v \in W^{1,2}(B_r)$,

$$\frac{1}{r}\int_{B_r} v^2 \leq C_2(n)\left(\int_{S_r} v^2 + r\int_{B_r} |\nabla v|^2\right),$$

we conclude that

$$\|\tilde{u}_{r_j}\|_{W^{1,2}(B_1)} < \infty.$$

□

Lemma 6.6. *Let $u \in \mathfrak{S}$ and suppose that $0 \in \Gamma(u)$. Assume that the hypothesis of Theorem 3.3 be satisfied. Given $r_j \to 0$, there exists a subsequence (which we will still denote by r_j) and for any $\alpha \in (0, 1/2)$ a function $\tilde{u}_0 \in C^{1,\alpha}_{loc}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$, such that $\tilde{u}_{r_j} \to \tilde{u}_0$ in $C^{1,\alpha}_{loc}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$. Such $\tilde{u}_0$ is a global solution of the Signorini problem (1.4)–(1.8) in $\mathbb{R}^n$ with $A \equiv I_n$, and we have $\tilde{u}_0 \neq 0$.*

Proof. We begin by observing that, as it was proved in [GS], we have $u \in C^{1,\frac{1}{2}}_{\rm loc}(B_1^\pm \cup B_1')$ with

$$\|u\|_{C^{1,\frac{1}{2}}(B^\pm_{1/2}\cup B'_{1/2})} \leq C(n, \lambda, Q, \|u\|_{W^{1,2}(B_1)}).$$

Given $r_j \searrow 0$, consider the sequence $\{\tilde{u}_{r_j}\}_{j\in\mathbb{N}}$. By Lemma 6.5 such sequence is uniformly bounded in $W^{1,2}(B_1)$. For any $\alpha \in (0, 1/2)$, by a standard diagonal process we obtain convergence in $C^{1,\alpha}_{\rm loc}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$ to a function $\tilde{u}_0$ of a subsequence of the functions $\tilde{u}_{r_j}$. Passing to the limit in (6.5) we conclude that such $\tilde{u}_0$ is a global solution to the Signorini problem (1.4)–(1.8) with $A \equiv I_n$. Clearly, $\tilde{u}_0$ is even in x_n. Finally, since by (6.4) we have

$$1 = \int_{S_1} \tilde{u}_r^2 \mu_r \to \int_{S_1} \tilde{u}_0^2,$$

we conclude that $\tilde{u}_0 \not\equiv 0$. □

Definition 6.7. We call the function $\tilde{u}_0$ in Lemma 6.6 a *Almgren blowup* of the function $u \in \mathfrak{S}$ at zero.

Proposition 6.8. *Let $u \in \mathfrak{S}$, $0 \in \Gamma(u)$, and suppose that the hypothesis of Theorem 3.3 be satisfied. Let $\tilde{u}_0$ be a Almgren blowup of u at zero. If $N(0^+) = \lim_{r\to 0^+} N(r)$ exists, then $\tilde{u}_0$ is a homogeneous function of degree $\kappa = N(0^+)$.*

Proof. Let $0 < r < \infty$ be fixed and consider a sequence r_j such that $\tilde{u}_{r_j} \to \tilde{u}_0$ as in Lemma 6.6. Since $\tilde{u}_{r_j} \to \tilde{u}_0$ in $C^{1,\alpha}_{loc}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$, we have $N_{L_{r_j}}(\tilde{u}_{r_j}, r) \to N_\Delta(\tilde{u}_0, r)$. On the other hand, by the fact that $N(0^+) = N_L(u, 0^+)$ exists, we have $N_L(u, rr_j) \to N_L(u, 0^+)$. Now, Lemma 6.4 gives $N_L(u, rr_j) = N_{L_{r_j}}(\tilde{u}_{r_j}, r)$. Passing to the limit as $j \to \infty$ in this equality, we infer

$$N_\Delta(\tilde{u}_0, r) \equiv N_L(u, 0^+), \quad 0 < r < \infty.$$

Since Almgren's frequency is constant and equal to κ if and only if the relevant function is homogeneous of degree κ, see [ACS], we conclude that $\tilde{u}_0$ must be homogeneous of degree $\kappa = N(0^+)$. □

We next analyze the asymptotic behavior of the homogeneous scalings (6.3).

Lemma 6.9. *Let $u \in \mathfrak{S}$, $0 \in \Gamma(u)$, and assume that the hypothesis of Theorem 3.3 be satisfied. Suppose that $N(0^+) \geq \kappa$. Given $r_j \to 0$, there exists a subsequence (which we will still denote by r_j) and for any $\alpha \in (0,1/2)$ a function $u_0 \in C^{1,\alpha}_{loc}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$, such that $u_{r_j} \to u_0$ in $C^{1,\alpha}_{loc}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$. Such u_0 is a global solution of the Signorini problem (1.4)–(1.8) in $\mathbb{R}^n$ with $A \equiv I_n$.*

Proof. We begin by observing that, under the given hypothesis, the conclusion of Lemma 4.1 is in force. Consider the family $\{u_{r_j}\}_{j\in\mathbb{N}}$. By (4.5) we have

$$\int_{B_1} u_{r_j}^2 = r_j^{-2\kappa} \int_{B_1} u(r_j x)^2 = r_j^{-(n+2\kappa)} \int_{B_{r_j}} u^2 \leq C_1.$$

Similarly, using (4.7) we find

$$\int_{B_1} |\nabla u_{r_j}|^2 = r_j^{2-2\kappa} \int_{B_1} |\nabla u(r_j x)|^2 = r_j^{-n+2-2\kappa} \int_{B_{r_j}} |\nabla u|^2 \leq \lambda^{-1} C^\star.$$

We conclude that $\{u_{r_j}\}_{j\in\mathbb{N}}$ is uniformly bounded in $W^{1,2}(B_1)$. Moreover, as proved in [GS], $u \in C^{1,\frac{1}{2}}_{\text{loc}}(B_1^\pm \cup B_1')$ with

$$\|u\|_{C^{1,\frac{1}{2}}(B^\pm_{1/2} \cup B'_{1/2})} \leq C(n, \lambda, Q, \|u\|_{W^{1,2}(B_1)}).$$

By a standard diagonal process, for any $\alpha \in (0,1/2)$, we obtain convergence in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$ of a subsequence of the functions u_{r_j} to a function u_0. Passing to the limit in (6.5), which also holds for u_r, we conclude that such u_0 is a global solution to the Signorini problem (1.4)–(1.8) with $A \equiv I_n$. Clearly, u_0 is even in x_n. □

Definition 6.10. We call the function u_0 in Lemma 6.9 a *homogeneous blowup* of $u \in \mathfrak{S}$ at zero.

Remark 6.11. We note that, unlike what happens for the Almgren blowups in Lemma 6.6, it is not guaranteed that a homogeneous blowup be nonzero.

The name homogeneous blowup is particularly justified by the following result that rests crucially on Theorem 5.3.

Proposition 6.12. *Let u_0 be a homogeneous blowup as in Lemma 6.9. Then, u_0 is a homogeneous function of degree $\kappa = N(0^+)$.*

Proof. Let $0 < r < R$. For a fixed r_j we integrate (5.1) in Theorem 5.3 over the interval $[r_j r, r_j R]$, obtaining:

$$\begin{aligned}
W_\kappa(r_jR,u) - W_\kappa(r_jr,u) + Cr_j(R-r) &\geq 2\int_{r_jr}^{r_jR} \frac{1}{t^{n+2\kappa}} \int_{S_t} \left(\frac{\langle A\nabla u, x\rangle}{\sqrt{\mu}} - \kappa\sqrt{\mu}u\right)^2 d\sigma dt \\
&= 2r_j \int_r^R \frac{1}{(r_js)^{n+2\kappa}} \int_{S_{r_js}} \left(\frac{\langle A\nabla u, x\rangle}{\sqrt{\mu}} - \kappa\sqrt{\mu}u\right)^2 d\sigma ds \\
&= 2r_j^n \int_r^R \frac{1}{(r_js)^{n+2\kappa}} \int_{S_s} \left(\frac{\langle A(r_jy)\nabla u(r_jy), r_jy\rangle}{\sqrt{\mu(r_jy)}} - \kappa\sqrt{\mu(r_jy)}u(r_jy)\right)^2 d\sigma ds \\
&= \frac{2}{r_j^{2\kappa}} \int_r^R \frac{1}{s^{n+2\kappa}} \int_{S_s} \left(\frac{\langle A_{r_j}(y)\nabla u_{r_j}(y), y\rangle}{\sqrt{\mu_{r_j}(y)}} r_j^\kappa - \kappa\sqrt{\mu_{r_j}(y)}u_{r_j}(y)r_j^\kappa\right)^2 \\
&= 2\int_{B_R\setminus B_r} \frac{1}{|y|^{n+2\kappa}} \left(\frac{\langle A_{r_j}(y)\nabla u_{r_j}(y), y\rangle}{\sqrt{\mu_{r_j}(y)}} - \kappa\sqrt{\mu_{r_j}(y)}u_{r_j}(y)\right)^2.
\end{aligned}$$

We want to take the limit as $r_j \to 0$ in the above inequality. By the second part of Theorem 5.3 we know that $W_\kappa(0+)$ exists (in fact, since we are assuming that $N(0^+) = \kappa$, by the second part of Lemma 5.2 we know that $W_\kappa(0+) = 0$). Now, the left-hand side of the above inequality goes to zero. Since $A(0) = I_n$, by (2) of Lemma 2.2 we know that $\mu_{r_j}(x) \to 1$ locally uniformly in x. From this, and the fact that u_{r_j} converges to u_0 in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$, letting $j \to \infty$ in the above inequality we infer that the latter converges to

$$0 \geq 2\int_{B_R\setminus B_r} \frac{1}{|y|^{n+2\kappa}} \left(\langle \nabla u_0, y\rangle - \kappa u_0\right)^2.$$

By the arbitrariness of $0 < r < R < \infty$ we conclude that u_0 is homogeneous of degree κ in $\mathbb{R}^n$. □

7. Some uniformity matters

We note that in all results in the above Sections 3–6 we have used in a crucial way the assumption that at the fixed free boundary point $0 \in \Gamma(u)$ the normalization $A(0) = I_n$ be in force. Since of course this is not necessarily the case at a generic point $x_0 \in \Gamma(u)$, we next discuss a change of coordinates which will allow us to deal with this problem while at the same time keeping some important matters of uniformity under control.

Given the ball $B_1 \subset \mathbb{R}^n$ and a function $u \in \mathfrak{S}$, suppose that $A(x_0)$ is not the identity matrix I_n for a given point $x_0 \in \Gamma(u) \subset B_1'$. We consider the affine transformation $T_{x_0} : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$T_{x_0}(x) = A(x_0)^{-1/2}(x - x_0). \tag{7.1}$$

T_{x_0} is a bijection from B_1 onto its image $\Omega_{x_0} = T_{x_0}(B_1)$, and we clearly have $T_{x_0}(x_0) = 0 \in \Omega_{x_0}$. We have the following

Lemma 7.1. *The transformation T_{x_0} maps the thin manifold into itself, i.e.,*

$$T_{x_0}(B_1') \subset M, \quad \textit{where} \quad M = \{(x', 0) \mid x' \in \mathbb{R}^{n-1}\}.$$

Proof. It suffices to show that a normal field to $T_{x_0}(B_1')$ is constant and parallel to e_n. Now, if $\tilde\eta(y)$ denotes a normal field on $T_{x_0}(B_1')$, it is easy to recognize that for $y \in T_{x_0}(B_1')$ we have

$$\tilde\eta(y) = A(x_0)^{1/2}\eta(T_{x_0}^{-1}(y)),$$

where $\eta(x)$ is a normal in $x \in B_1'$. Since we can take $\eta(x) \equiv e_n$ on B_1', we conclude that

$$\tilde\eta(y) \equiv A(x_0)^{1/2}e_n, \quad y \in T_{x_0}(B_1')$$

is a normal field on $T_{x_0}(B_1')$. Hence, $\tilde\eta(y)$ is constant on $T_{x_0}(B_1')$. Now the fact that $A(x_0)$ has the form (1.3) implies that

$$A(x_0)e_n = a_{nn}(x_0)e_n, \quad A(x_0)e_i = \begin{pmatrix} a_{1i}(x_0) \\ a_{2i}(x_0) \\ \cdots \\ a_{n-1,i}(x_0) \\ 0 \end{pmatrix}, \quad i = 1, \ldots, n-1.$$

Therefore, e_n is an eigenvector of $A(x_0)$ with corresponding (positive) eigenvalue $\lambda_n(x_0) = a_{nn}(x_0)$. Since the eigenvalues of $A(x_0)^{1/2}$ are the square roots of those of $A(x_0)$, and the eigenvectors of $A(x_0)^{1/2}$ are the same as those of $A(x_0)$, we conclude that $\sqrt{\lambda_n(x_0)} = \sqrt{a_{nn}(x_0)}$ is an eigenvalue of $A(x_0)^{1/2}$, and in fact $\tilde\eta(y) = A(x_0)^{1/2}e_n = \sqrt{a_{nn}(x_0)}e_n$. We conclude that $T_{x_0}(B_1') \subset B_1'$. □

For a given function $u : B_1 \to \mathbb{R}$, and a given point $x_0 \in B_1'$, we consider the function $u_{x_0} : \Omega_{x_0} \to \mathbb{R}$ defined by

$$u_{x_0}(y) = u \circ T_{x_0}^{-1}(y) = u(x_0 + A^{1/2}(x_0)y), \quad y \in \Omega_{x_0}. \tag{7.2}$$

For a given matrix-valued function A defined in B_1 we consider A_{x_0} defined in Ω_{x_0} as follows

$$A_{x_0}(y) = A^{-1/2}(x_0)A(x_0 + A^{1/2}(x_0)y)A^{-1/2}(x_0). \tag{7.3}$$

Lemma 7.2. *Suppose that $u \in \mathfrak{S}$, and that $x_0 \in \Gamma(u)$. Let $R_0 = \operatorname{dist}(x_0, S_1)$. Then, the function u_{x_0} is even in x_n and satisfies the Signorini conditions* (1.4)–(1.8) *with respect to the matrix A_{x_0} in the ball $B_{\sqrt{\lambda}R_0}(0)$. In particular, one has in $B^+_{\sqrt{\lambda}R_0}(0) \cup B^-_{\sqrt{\lambda}R_0}(0)$*

$$L_{x_0}u_{x_0} := \operatorname{div}(A_{x_0}(y)\nabla u_{x_0}) = 0. \tag{7.4}$$

Furthermore, the matrix A_{x_0} satisfies

$$\lambda^2|\xi|^2 \;\le\; \langle A_{x_0}(y)\xi, \xi\rangle \;\le\; \lambda^{-2}|\xi|^2, \tag{7.5}$$

for any $y \in B_{\sqrt{\lambda}R_0}(0)$, and any $\xi \in \mathbb{R}^n$. Also, it satisfies (2.3)*. Finally, the entries of A_{x_0} satisfy conditions* (1.3)*, and moreover $A_{x_0}(0) = I_n$.*

Proof. In a standard way one verifies that if u is a weak solution to $Lu = \operatorname{div}(A(x)\nabla u) = 0$ in $B_1^\pm$, then u_{x_0} is a weak solution in $\Omega^\pm_{x_0} = T_{x_0}(B_1^\pm)$ to

$$L_{x_0}u_{x_0} = \operatorname{div}(A_{x_0}(y)\nabla u_{x_0}) = 0. \tag{7.6}$$

Notice that $T_{x_0}(x_0) = 0 \in \Omega_{x_0}$, and by construction we have $A_{x_0}(0) = I_n$.

We note explicitly that in passing from the matrix A in B_1 to the matrix A_{x_0} in Ω_{x_0} the uniform bounds on the ellipticity change from λ to λ^2. We have in fact that (7.5) holds for every $y \in \Omega_{x_0}$ and any $\xi \in \mathbb{R}^n$. Moreover, hypothesis (2.2) implies that for every $x_0 \in B_1'$ and $x, p \in \mathbb{R}^n$

$$\begin{aligned} \lambda^{1/2}|x-p| \le |A^{1/2}(x_0)(x-p)| \le \lambda^{-1/2}|x-p|, \\ \lambda^{1/2}|x-p| \le |A^{-1/2}(x_0)(x-p)| \le \lambda^{-1/2}|x-p|. \end{aligned} \tag{7.7}$$

We can rewrite the second inequality in (7.7) in the following way

$$\lambda^{1/2}|x-p| \le |T_{x_0}(x) - T_{x_0}(p)| \le \lambda^{-1/2}|x-p|, \tag{7.8}$$

or, equivalently,

$$B_{\sqrt{\lambda}r}(T_{x_0}(p)) \subset T_{x_0}(B_r(p)) \subset B_{\frac{r}{\sqrt{\lambda}}}(T_{x_0}(p)), \tag{7.9}$$

for any $p \in \mathbb{R}^n$ and $r > 0$. In particular, if we take $p = x_0$, and recalling that $T_{x_0}(x_0) = 0$, we have from (7.9)

$$B_{\sqrt{\lambda}r}(0) \subset T_{x_0}(B_r(x_0)) \subset B_{\frac{r}{\sqrt{\lambda}}}(0). \tag{7.10}$$

If we take $R_0 = \operatorname{dist}(x_0, S_1)$, we conclude from (7.10) that the function u_{x_0} satisfies the equation (7.4) in the half-balls $B^{\pm}_{\sqrt{\lambda}R_0}(0)$.

Finally, we note that the matrix-valued function $y \to A_{x_0}(y)$ satisfies in Ω_{x_0} an assumption similar to (2.3). In fact, from (7.7) and (2.3) we have

$$\|A_{x_0}(y) - A_{x_0}(y')\| \le \lambda^{-3/2} Q|y-y'|, \quad y, y' \in \Omega_{x_0}. \tag{7.11}$$

Given any $x_0 \in \Gamma(u)$, we can now move x_0 to the origin by considering the function $u_{x_0} : \Omega_{x_0} \to \mathbb{R}$ defined as in (7.2). From what we have shown u_{x_0} satisfies (7.4) in the half balls $B^{\pm}_{\sqrt{\lambda}R_0}(0)$. If we denote by $\Gamma(u_{x_0})$ the free boundary of u_{x_0} in the ball $B_{\sqrt{\lambda}R_0}(0)$, then we have

$$\Gamma(u_{x_0}) \subset B'_{\sqrt{\lambda}R_0}(0) = \{(y', 0) \in \mathbb{R}^n \mid |y'| < \sqrt{\lambda}R_0\}.$$

Moreover, we claim that $u_{x_0}(y', y_n)$ is even in y_n in the ball $B_{\sqrt{\lambda}R_0}(0)$, i.e.,

$$u_{x_0}(y', -y_n) = u_{x_0}(y', y_n). \tag{7.12}$$

This can be easily seen as follows. We write $(y', y_n) = (y', 0) + y_n e_n$. Then,

$$u_{x_0}(y', -y_n) = u(x_0 + A^{1/2}(x_0)(y', 0) - y_n A^{1/2}(x_0)e_n) = u(T_{x_0}^{-1}(y', 0) - \sqrt{a_{nn}(x_0)}y_n e_n).$$

Since $x_0 = (x_0', 0)$, and we have shown that T_{x_0}, and therefore $T_{x_0}^{-1}$, maps M onto M, from the evenness of u in x_n we conclude that

$$u_{x_0}(y', -y_n) = u(T_{x_0}^{-1}(y', 0) - \sqrt{a_{nn}(x_0)}y_n e_n) = u(T_{x_0}^{-1}(y', 0) + \sqrt{a_{nn}(x_0)}y_n e_n) = u_{x_0}(y', y_n).$$

This proves (7.12). The proof of the remaining part of the lemma is left as an exercise to the reader. □

Having dealt with these matters of uniformity, we introduce the *Almgren scalings* and the homogeneous scalings at an arbitrary point $x_0 \in \Gamma(u)$.

Definition 7.3. Let $u \in \mathfrak{S}$ and suppose that $x_0 \in \Gamma(u)$. With u_{x_0} as in (7.2) above, we define

$$\tilde{u}_{x_0,r}(x) = \frac{u_{x_0}(rx)}{d_{x_0,r}}, \quad \text{where} \quad d_{x_0,r} = \left(\frac{1}{r^{n-1}} H_{x_0}(r)\right)^{1/2}. \tag{7.13}$$

The *homogeneous scalings* of u at x_0 are defined in the following way:

$$u_{x_0,r}(x) = \frac{u_{x_0}(rx)}{r^\kappa}. \tag{7.14}$$

When $x_0 = 0$, we simply write $\tilde{u}_r$ and u_r.

Notice that, if we let

$$\mu_{x_0,r}(x) = \frac{\langle A_{x_0}(rx)x, x\rangle}{|x|^2},$$

then we have the following normalization:

$$\int_{S_1} \tilde{u}_{x_0,r}^2 \ \mu_{x_0,r} = 1.$$

8. Singular set of the free boundary

Let $u \in \mathfrak{S}$. In this section we introduce those free boundary points of u that constitute the main focus of this paper and prove a characterization result.

Definition 8.1. Let $u \in \mathfrak{S}$ and let $\kappa \geq 3/2$. We say that $x_0 \in \Gamma_\kappa(u)$ if $x_0 \in \Gamma(u)$ and $N_{x_0}(u_{x_0}, 0+) = \kappa$, where $N_{x_0}(u_{x_0}, r) := N_{L_{x_0}}(u_{x_0}, r)$.

Definition 8.2 (Singular points)**.** Let $u \in \mathfrak{S}$. We say that $x_0 \in \Gamma(u)$ is a *singular point* of the free boundary if

$$\lim_{r\to 0+} \frac{\mathcal{H}^{n-1}(\Lambda(u_{x_0}) \cap B'_r)}{\mathcal{H}^{n-1}(B'_r)} = 0.$$

We denote with $\Sigma(u)$ the subset of singular points of $\Gamma(u)$. We also denote

$$\Sigma_\kappa(u) = \Sigma(u) \cap \Gamma_\kappa(u).$$

Notice that in terms of the scalings $\tilde{u}_{x_0,r}$, the condition $0 \in \Sigma(u)$ is equivalent to

$$\lim_{r\to 0+} \mathcal{H}^{n-1}(\Lambda(\tilde{u}_{x_0,r}) \cap B'_1) = 0. \tag{8.1}$$

Before stating the next result we introduce a definition.

Definition 8.3. In what follows we will indicate with $\mathfrak{P}^+_\kappa(\mathbb{R}^n)$ the class of all nonzero homogeneous polynomials p_κ of degree κ in $\mathbb{R}^n$, such that:

$$\Delta p_\kappa = 0, \quad p_\kappa(x', 0) \geq 0, \quad p_\kappa(x', -x_n) = p_\kappa(x', x_n).$$

Theorem 8.4 (Characterization of singular points)**.** *Let $u \in \mathfrak{S}$ with $0 \in \Gamma_\kappa(u)$ for $\kappa > 3/2$. The following statements are equivalent:*

(i) $0 \in \Sigma_\kappa(u)$.

(ii) *Any Almgren blowup of u at the origin (as in Lemma 6.6), $\tilde{u}_0$, is a nonzero homogeneous polynomial $p_\kappa \in \mathfrak{P}^+_\kappa(\mathbb{R}^n)$.*

(iii) $\kappa = 2m$, *for some $m \in \mathbb{N}$.*

Proof. It follows as the proof of Theorem 1.3.2 of [GP], with minor changes.
(i) $\Rightarrow$ (ii) *Step 1: $\tilde{u}_0$ is harmonic.* By (6.5) for $0 < r < 1$ and any $\eta \in C^\infty_0(B_1)$ we find

$$\int_{B_1} \langle A_r \nabla \tilde{u}_r, \nabla \eta\rangle = -2 \int_{B'_1 \cap \Lambda(\tilde{u}_r)} (a_{nn})_r D^+_n \tilde{u}_r \eta. \tag{8.2}$$

By Lemma 6.6 $|\nabla \tilde{u}_r|$ are uniformly bounded in B_1. This fact and (8.1) allow to conclude that

$$\lim_{r\to 0+} \int_{B_1'\cap\Lambda(\tilde{u}_r)} (a_{nn})_r D_n^+ \tilde{u}_r \eta = 0. \tag{8.3}$$

On the other hand, since $A(0) = I_n$, we infer that

$$\int_{B_1} \langle A_r \nabla \tilde{u}_r, \nabla \eta\rangle \to \int_{B_1} \langle \nabla \tilde{u}_0, \nabla \eta\rangle.$$

We conclude that $\tilde{u}_0$ is weakly harmonic in B_1, and therefore by the Caccioppoli-Weyl lemma it is a classical harmonic function in B_1.

Step 2: $\tilde{u}_0$ is a polynomial. Since by Proposition 6.8 $\tilde{u}_0$ is homogeneous of degree κ, it can be extended to all of $\mathbb{R}^n$ and such extension will be harmonic. By homogeneity, $\tilde{u}_0$ has at most polynomial growth at infinity. Using Louville's Theorem, we conclude that $\tilde{u}_0$ must be a polynomial of degree κ.

The implications (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (ii), and (ii) $\Rightarrow$ (i) follow as in Theorem 1.3. in [GP], and we refer to that source. □

Similarly, we can derive more information on homogeneous blowups around singular points.

Lemma 8.5. *Let $u \in \mathfrak{S}$ with $0 \in \Sigma_\kappa(u)$. Then any homogeneous blowup of u at the origin (as in Lemma 6.9) is a homogeneous polynomial $p_\kappa \in \mathfrak{P}_\kappa^+(\mathbb{R}^n) \cup \{0\}$.*

Proof. We notice that (8.2) still holds with u_r instead of $\tilde{u}_r$. Moreover, as shown in the proof of Lemma 6.9, $\{u_r\}_{r<1}$ is uniformly bounded in $W^{1,2}(B_1)$, and by assumption

$$\lim_{r\to 0+} \mathcal{H}^{n-1}(\Lambda(u_r)\cap B_1') = 0.$$

The proof then follows exactly as in (i) $\Rightarrow$ (ii) in Theorem 8.4. □

Remark 8.6. Notice that we still cannot conclude that p_κ is nonzero. This will follow from Lemma 10.3 below.

9. A one-parameter family of Monneau type monotonicity formulas

The objective of this section is to establish a generalization of the one-parameter Monneau type monotonicity formulas that were obtained in [GP] for solutions of the Signorini problem for the Laplacian. Since our main result will appear as a perturbation of the constant coefficient one, in what follows we will consider harmonic polynomials p_κ in $\mathbb{R}^n$ which are homogeneous of degree κ and such that $p_\kappa(x', 0) \geq 0$. For a function p we define

$$\Psi_p(r) = \frac{1}{r^{n-2+2\kappa}} \int_{B_r} |\nabla p|^2 - \frac{\kappa}{r^{n-1+2\kappa}} \int_{S_r} p^2.$$

Lemma 9.1. *For any harmonic polynomial p_κ which is homogeneous of degree κ, we have for every $r > 0$*

$$\Psi_{p_\kappa}(r) = 0.$$

Proof. Since $\Delta p_\kappa = 0$ we have $\Delta p_\kappa^2 = 2|\nabla p_\kappa|^2$. Integrating by parts and using the fact that p_κ is κ homogeneous, we thus find

$$\Psi_{p_\kappa}(r) = \frac{1}{r^{n-1+2\kappa}} \int_{S_r} p_\kappa \langle \nabla p_\kappa, \nu \rangle - \frac{\kappa}{r^{n-1+2\kappa}} \int_{S_r} p_\kappa^2 = 0. \tag{9.1}$$

□

Definition 9.2. Let $u \in \mathfrak{S}$ and let $0 \in \Gamma_\kappa(u)$. For any $p_\kappa \in \mathfrak{P}_\kappa^+(\mathbb{R}^n)$ we define

$$M_\kappa(u, p_\kappa, r) = M_\kappa(r) = \frac{1}{r^{n-1+2\kappa}} \int_{S_r} (u - p_\kappa)^2 \mu,$$

where μ is the conformal factor introduced in (2.5).

The following theorem is the main result of this section.

Theorem 9.3 (Monneau type monotonicity formula)**.** *Let $u \in \mathfrak{S}$ and assume that $0 \in \Gamma_\kappa(u)$. Suppose that the hypothesis of Theorem 3.3 be satisfied. Then there exists a universal constant $\tilde{C} > 0$ such that*

$$\frac{d}{dr}\left(M_\kappa(r) + \tilde{C} r\right) \geq \frac{2 W_\kappa(r)}{r}. \tag{9.2}$$

Proof. Let $w = u - p_\kappa$, so that $M_\kappa(r) = \frac{1}{r^{n-1+2\kappa}} \int_{S_r} w^2 \mu$. Then, as in Lemma 4.4 in [GS], we find

$$M_\kappa'(r) = -\frac{n-1+2\kappa}{r^{n+2\kappa}} \int_{S_r} w^2 \mu + \frac{1}{r^{n-1+2\kappa}} \left[2 \int_{S_r} w \langle A \nabla w, \nabla r \rangle + \int_{S_r} w^2 L|x| \right].$$

At this point we invoke (2.4) in Lemma 2.1, (2) in Lemma 2.2 and (4.4) in Lemma 4.2 to obtain from the latter equation

$$M_\kappa'(r) = \frac{2}{r^{n+2\kappa}} \int_{S_r} w \langle A \nabla w, x \rangle - \frac{2\kappa}{r^{n+2\kappa}} \int_{S_r} w^2 \mu + O(1). \tag{9.3}$$

We next consider

$$W_\kappa(r, u) = W_\kappa(r) = \frac{1}{r^{n-2+2\kappa}} \int_{B_r} \langle A \nabla u, \nabla u \rangle - \frac{\kappa}{r^{n-1+2\kappa}} \int_{S_r} u^2 \mu,$$

see Definition 5.1. Using (9.1), and recalling that we are assuming $A(0) = I_n$, we can write

$$\begin{aligned}
W_\kappa(r) &= W_\kappa(r) - \Psi_{p_\kappa}(r) \\
&= \frac{1}{r^{n-2+2\kappa}} \int_{B_r} \langle A \nabla u, \nabla u \rangle - \frac{\kappa}{r^{n-1+2\kappa}} \int_{S_r} u^2 \mu - \frac{1}{r^{n-2+2\kappa}} \int_{B_r} |\nabla p_\kappa|^2 + \frac{\kappa}{r^{n-1+2\kappa}} \int_{S_r} p_\kappa^2 \\
&= \frac{1}{r^{n-2+2\kappa}} \int_{B_r} (\langle A \nabla u, \nabla u \rangle - \langle A \nabla p_\kappa, \nabla p_\kappa \rangle) + \frac{1}{r^{n-2+2\kappa}} \int_{B_r} \langle (A(x) - A(0)) \nabla p_\kappa, \nabla p_\kappa \rangle \\
&\quad - \frac{\kappa}{r^{n-1+2\kappa}} \int_{S_r} (u^2 - p_\kappa^2) \mu + \frac{\kappa}{r^{n-1+2\kappa}} \int_{S_r} p_\kappa^2 (1 - \mu).
\end{aligned}$$

Since p_κ is homogeneous of degree κ, we have

$$\frac{1}{r^{n-2+2\kappa}} \int_{B_r} \langle (A(x) - A(0)) \nabla p_\kappa, \nabla p_\kappa \rangle = O(r).$$

On the other hand, the estimate (2) in Lemma 2.2 and the homogeneity of p_κ again give

$$\frac{\kappa}{r^{n-1+2\kappa}} \int_{S_r} p_\kappa^2 (1 - \mu) = O(r).$$

We conclude that

$$\begin{aligned} W_\kappa(r) &= \frac{1}{r^{n-2+2\kappa}} \int_{B_r} (\langle A\nabla u, \nabla u\rangle - \langle A\nabla p_\kappa, \nabla p_\kappa\rangle) \\ &\quad - \frac{\kappa}{r^{n-1+2\kappa}} \int_{S_r} (u^2 - p_\kappa^2)\mu + O(r). \end{aligned}$$

It is now a simple computation to verify that the difference of the two integrals in the right-hand side of the latter equality can be expressed in the following form

$$\begin{aligned} W_\kappa(r) &= \frac{1}{r^{n-2+2\kappa}} \int_{B_r} (\langle A\nabla w, \nabla w\rangle + 2\langle A\nabla w, \nabla p_\kappa\rangle) \\ &\quad - \frac{\kappa}{r^{n-1+2\kappa}} \int_{S_r} (w^2 + 2wp_\kappa)\mu + O(r). \end{aligned} \tag{9.4}$$

Using the divergence theorem we find

$$\int_{B_r} \langle A\nabla w, \nabla p_\kappa\rangle = \int_{S_r} w\langle A\nabla p_\kappa, \nu\rangle - \int_{B_r} wLp_\kappa.$$

Using the harmonicity of p_κ and letting $B(x) = A(x) - A(0)$, we have

$$Lp_\kappa = \operatorname{div}(A(x)\nabla p_\kappa) = \Delta p_k + \operatorname{div}(B(x)\nabla p_\kappa) = D_i(b_{ij})D_j p_\kappa + b_{ij}D_{ij}p_\kappa.$$

By (2.3), and by the fact that p_κ is homogeneous of degree κ, we conclude that for a.e. $x \in B_r$ we have

$$Lp_\kappa(x) = O(|x|)|x|^{\kappa-2}.$$

This fact and (4.4) in Lemma 4.2 allow to conclude that

$$\int_{B_r} wLp_\kappa = O(r^{n-1+2\kappa}). \tag{9.5}$$

Substituting (9.5) into (9.4), we find

$$\begin{aligned} W_\kappa(r) &= \frac{1}{r^{n-2+2\kappa}} \int_{B_r} \langle A\nabla w, \nabla w\rangle + \frac{2}{r^{n-1+2\kappa}} \int_{S_r} w\langle A\nabla p_\kappa, x\rangle \\ &\quad - \frac{\kappa}{r^{n-1+2\kappa}} \int_{S_r} (w^2 + 2wp_\kappa)\mu + O(r). \end{aligned} \tag{9.6}$$

We now integrate by parts in the first integral in the right-hand side of (9.6). Using the properties of p_κ and the Signorini conditions (1.4)–(1.8), we obtain

$$\begin{aligned} \frac{1}{r^{n-2+2\kappa}} \int_{B_r} \langle A\nabla w, \nabla w\rangle &= \frac{1}{r^{n-2+2\kappa}} \int_{B_r} wLp_\kappa + \frac{1}{r^{n-2+2\kappa}} \int_{S_r} w\langle A\nabla w, \nu\rangle + \\ &\quad + \frac{1}{r^{n-2+2\kappa}} \int_{B_r'} w(\langle A\nabla w, \nu_+\rangle + \langle A\nabla w, \nu_-\rangle) \\ &= -\frac{1}{r^{n-2+2\kappa}} \int_{B_r'} p_\kappa(\langle A\nabla u, \nu_+\rangle + \langle A\nabla u, \nu_-\rangle) \\ &\quad + \frac{1}{r^{n-2+2\kappa}} \int_{S_r} w\langle A\nabla w, \nu\rangle + O(r), \end{aligned}$$

where in the last equality we have used (9.5) again. Recalling now (1.6) and the fact that $p_\kappa(x', 0) \geq 0$, we have

$$-\frac{1}{r^{n-2+2\kappa}} \int_{B_r'} p_\kappa(\langle A\nabla u, \nu_+\rangle + \langle A\nabla u, \nu_-\rangle) \leq 0.$$

In conclusion,

$$\frac{1}{r^{n-2+2\kappa}}\int_{B_r}\langle A\nabla w,\nabla w\rangle \ \le\ \frac{1}{r^{n-1+2\kappa}}\int_{S_r} w\langle A\nabla w,x\rangle+O(r).$$

Substituting this in (9.6) above and recalling (9.3), we conclude

$$\begin{aligned}W_\kappa(r)\le{}&\frac{1}{r^{n-1+2\kappa}}\int_{S_r} w\langle A\nabla w,x\rangle+\frac{2}{r^{n-1+2\kappa}}\int_{S_r} w\langle A\nabla p_\kappa,x\rangle\\&-\frac{\kappa}{r^{n-1+2\kappa}}\int_{S_r}(w^2+2wp_\kappa)\mu+O(r)\\={}&\frac{rM_\kappa'(r)}{2}+\frac{\kappa}{r^{n-1+2\kappa}}\int_{S_r} w^2\mu+\frac{2}{r^{n-1+2\kappa}}\int_{S_r} w\langle A\nabla p_\kappa,x\rangle\\&-\frac{\kappa}{r^{n-1+2\kappa}}\int_{S_r}(w^2+2wp_\kappa)\mu+O(r)\\={}&\frac{rM_\kappa'(r)}{2}+\frac{2}{r^{n-1+2\kappa}}\int_{S_r} w\langle A\nabla p_\kappa,x\rangle\\&-\frac{2\kappa}{r^{n-1+2\kappa}}\int_{S_r} wp_\kappa\mu+O(r).\end{aligned}$$

Finally, since $\langle\nabla p_\kappa,x\rangle-\kappa p_\kappa\equiv 0$, using again the fact that $A(0)=I_n$, (2.3), (2) in Lemma 2.2 and the estimate (4.4) in Lemma 4.2, one verifies that

$$\frac{2}{r^{n-1+2\kappa}}\int_{S_r} w\langle A\nabla p_\kappa,x\rangle-\frac{2\kappa}{r^{n-1+2\kappa}}\int_{S_r} wp_\kappa\mu=O(r).$$

In conclusion, we have proved that

$$W_\kappa(r)\le\frac{rM_\kappa'(r)}{2}+O(r).$$

This estimate gives, for a universal constant $\tilde C>0$ (depending also on κ)

$$M_\kappa'(r)\ge\frac{2W_\kappa(r)}{r}-\tilde C.$$

This implies that

$$\frac{d}{dr}\left(M_\kappa(r)+\tilde Cr\right)\ge\frac{2W_\kappa(r)}{r},$$

thus proving the theorem. □

We now draw an important consequence of Theorem 9.3.

Corollary 9.4. *Under the assumptions of Theorem 9.3 we have*

$$\frac{d}{dr}\left(M_\kappa(r)+C^\star r\right)\ge 0, \tag{9.7}$$

where $C^\star$ is a universal constant. In particular, the limit $M_\kappa(0^+)=\lim_{r\to 0}M_\kappa(r)$ exists.

Proof. By the second part of Lemma 5.2 the assumption $0\in\Gamma_\kappa(u)$ implies that $W_\kappa(0^+)=0$. By Theorem 5.3 there exists a constant $C>0$ such that $W_\kappa(r)+Cr$ is monotone nondecreasing. We infer that $W_\kappa(r)+Cr\ge W_\kappa(0^+)=0$. This implies $W_\kappa(r)\ge -Cr$. Combining this estimate with (9.2) in Theorem 9.3, we obtain

$$\frac{d}{dr}\left(M_\kappa(r)+\tilde Cr\right)\ge -2C.$$

The desired conclusion now immediately follows. □

10. Nondegeneracy

In this section we use the Almgren and Monneau monotonicity formulas to prove a non-degeneracy property, Lemma 10.2 below. This in turn allows us to prove the uniqueness of homogeneous blowups, and that such blowup cannot vanish identically. We further prove the the continuos dependence of the blowups. We start with a lower bound on $H(r)$ which is used to prove the nondegeneracy property.

Lemma 10.1. *Let $u \in \mathfrak{S}$ with $0 \in \Gamma_\kappa(u)$. Then*

$$r\frac{d}{dr}\log H(r) - (n-1+2\kappa) = 2(N(r)-\kappa) + O(r) \to 0, \quad \text{as } r \to 0^+. \tag{10.1}$$

In particular, for every $\varepsilon > 0$ there exist $r_\varepsilon \in (0,1)$ and a universal constant $C_\varepsilon > 0$ (depending also on u), such that for every $0 < r < r_\varepsilon$ one has

$$H(r) \ge C_\varepsilon r^{n-1+2\kappa+\varepsilon}. \tag{10.2}$$

Proof. By (4.2) we have

$$\frac{d}{dr}\log\frac{H(r)}{r^{n-1}} = 2\frac{N(r)}{r} + O(1),$$

for a.e. $0 < r < 1$. From this formula, and the fact that $N(0^+) = \kappa$, we immediately obtain (10.1). From (10.1) we see that for every $\varepsilon > 0$ there exists $r_\varepsilon > 0$ small such that

$$r\frac{d}{dr}\log H(r) \le 2\kappa + n - 1 + \varepsilon, \quad 0 < r < r_\varepsilon.$$

Integrating from r to r_ε, this gives

$$\frac{H(r_\varepsilon)}{H(r)} \le \left(\frac{r_\varepsilon}{r}\right)^{2\kappa+n-1+\varepsilon},$$

from which we conclude, with $C_\varepsilon = H(r_\varepsilon)/r_\varepsilon^{n-1+2\kappa+\varepsilon}$, that

$$H(r) \ge C_\varepsilon r^{n-1+2\kappa+\varepsilon}.$$

□

Lemma 10.2 (Nondegeneracy)**.** *Let $u \in \mathfrak{S}$, with $0 \in \Sigma_\kappa(u)$, and suppose that the hypothesis of Theorem 3.3 be satisfied. Then there exist universal $c > 0$ and $0 < r_0 < 1$, possibly depending on u, such that for $0 < r < r_0$ one has*

$$\sup_{S_r} |u(x)| \ge c r^\kappa. \tag{10.3}$$

Proof. We argue by contradiction and suppose that (10.3) does not hold. Then, there exists a sequence $r_j \to 0$ such that

$$\frac{\sup_{S_{r_j}} |u(x)|}{{r_j}^\kappa} \to 0.$$

This implies, in particular,

$$\frac{d_{r_j}}{{r_j}^\kappa} = \left(\frac{1}{r_j^{n-1+2\kappa}}\int_{S_{r_j}} u^2\mu\right)^{\frac{1}{2}} = o(1). \tag{10.4}$$

Consider now the sequence of Almgren scalings $\tilde{u}_{r_j}(x) = \frac{u(r_j x)}{d_{r_j}}$, $j \in \mathbb{N}$. From Lemma 6.9 and Proposition 6.12 above (see also (ii) of Theorem 8.4), we infer the existence of a nonzero polynomial $q_\kappa \in \mathfrak{P}_\kappa^+(\mathbb{R}^n)$, such that $\tilde{u}_{r_j} \to q_\kappa$ on $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$ as $j \to \infty$. Corollary 9.4 implies that the limit $M_\kappa(u, q_\kappa, 0^+)$ exists. Therefore, we can use the sequence r_j to compute such limit, i.e.,

$$M_\kappa(u, q_\kappa, 0^+) = \lim_{j\to\infty} M_\kappa(u, q_\kappa, r_j).$$

Definition 9.2 gives

$$M_\kappa(u, q_\kappa, r_j) = \frac{1}{r_j^{n-1+2\kappa}} \int_{S_{r_j}} (u - q_\kappa)^2 \mu = \frac{1}{r_j^{n-1+2\kappa}} \int_{S_{r_j}} (u^2 - 2uq_\kappa + q_\kappa^2)\mu.$$

Now, (10.4) gives

$$\frac{1}{r_j^{n-1+2\kappa}} \int_{S_{r_j}} u^2 \mu \to 0.$$

On the other hand, Lebesgue dominated convergence theorem gives

$$\lim_{j\to\infty} \int_{S_1} q_\kappa(y)^2 \mu(r_j y) = \mu(0) \int_{S_1} q_\kappa^2 = \int_{S_1} q_\kappa^2 < \infty.$$

We infer from this that

$$\begin{aligned} 0 \le & \frac{1}{r_j^{n-1+2\kappa}} \int_{S_{r_j}} |uq_\kappa|\mu \le \frac{1}{r_j^{n-1+2\kappa}} \left(\int_{S_{r_j}} u^2\mu\right)^{\frac12} \left(\int_{S_{r_j}} q_\kappa^2\mu\right)^{\frac12} \\ = & \left(\frac{1}{r_j^{n-1+2\kappa}} \int_{S_{r_j}} u^2\mu\right)^{\frac12} \left(\int_{S_1} q_\kappa(y)^2 \mu(r_j y)\right)^{\frac12} \to 0. \end{aligned}$$

In conclusion, we have

$$M_\kappa(u, q_\kappa, 0^+) = \lim_{j\to\infty} M_\kappa(u, q_\kappa, r_j) = \lim_{j\to\infty} \frac{1}{r_j^{n-1+2\kappa}} \int_{S_{r_j}} q_\kappa^2 \mu = \int_{S_1} q_\kappa^2. \tag{10.5}$$

By (10.5) and the homogeneity of q_κ we infer that for every $r \in (0, 1)$ we have

$$M_\kappa(u, q_\kappa, 0^+) = \frac{1}{r^{n-1+2\kappa}} \int_{S_r} q_\kappa^2. \tag{10.6}$$

Since according to Corollary 9.4 the function $r \to M_\kappa(r) + C^\star r$ is monotone nondecreasing, we have that

$$\frac{1}{r^{n-1+2\kappa}} \int_{S_r} (u - q_\kappa)^2 \mu + C^\star r \ge M_\kappa(u, q_\kappa, 0^+) = \frac{1}{r^{n-1+2\kappa}} \int_{S_r} q_\kappa^2,$$

where in the last equality we have used (10.6). Equivalently, we have

$$\frac{1}{r^{n-1+2\kappa}} \int_{S_r} (u^2 - 2uq_\kappa)\mu + \frac{1}{r^{n-1+2\kappa}} \int_{S_r} q_\kappa^2 \mu \ge -C^\star r + \frac{1}{r^{n-1+2\kappa}} \int_{S_r} q_\kappa^2. \tag{10.7}$$

Recalling that $\tilde{u}_r(x) = \frac{u(rx)}{d_r}$, (10.7) is equivalent to

$$\frac{1}{r^{2\kappa}} \int_{S_1} (d_r^2 \tilde{u}_r^2 - 2d_r r^\kappa \tilde{u}_r q_\kappa)\mu(rx) \ge -C^\star r + \int_{S_1} q_\kappa^2 (1 - \mu(rx)),$$

i.e.,

$$\int_{S_1} \left(\frac{d_r}{r^\kappa} \tilde{u}_r^2 - 2\tilde{u}_r q_\kappa\right) \mu(rx) \ge -C^\star \frac{r^{\kappa+1}}{d_r} + \frac{r^\kappa}{d_r} \int_{S_1} q_\kappa^2 (1 - \mu(rx)). \tag{10.8}$$

We claim that the right-hand side goes to 0 as $r \to 0$. Indeed, by the definition (6.1) of d_r and (10.2) in Lemma 10.1, for any $\varepsilon \in (0,1)$ there exists $r_\varepsilon, C_\varepsilon > 0$ such that for $0 < r < r_\varepsilon$ we have,

$$\frac{r^{2\kappa+2}}{d_r^2} = \frac{r^{2\kappa+2+n-1}}{H(r)} \le \frac{r^{2\kappa+n+1}}{C_\varepsilon r^{2\kappa+n-1+\varepsilon}} = \frac{r^{2-\varepsilon}}{C_\varepsilon}.$$

We infer that $\frac{r_j^{\kappa+1}}{d_{r_j}} \to 0$ as $j \to \infty$. By (ii) in Lemma 2.2 above we obtain for a universal constant $C > 0$

$$\left| \int_{S_1} q_\kappa^2 (1 - \mu(rx)) \right| \le \sup_{x \in S_1} |1 - \mu(rx)| \int_{S_1} q_\kappa^2 \le Cr \int_{S_1} q_\kappa^2.$$

Therefore, we have as $j \to \infty$

$$\left| \frac{r_j^\kappa}{d_{r_j}} \int_{S_1} q_\kappa^2 (1 - \mu(r_j x)) \right| \le C \frac{r_j^{\kappa+1}}{d_{r_j}} \int_{S_1} q_\kappa^2 \to 0.$$

In conclusion, if we let $r = r_j \to 0$ in (10.8) we obtain

$$-\int_{S_1} q_\kappa^2 \ge 0.$$

Since $q_\kappa \not\equiv 0$, we have thus reached a contradiction. □

With the nondegeneracy in hands we are finally ready to prove that, if zero is a singular point, then the homogeneous blowups of u at zero cannot vanish indentically.

Lemma 10.3. *Let $u \in \mathfrak{S}$, $0 \in \Sigma_\kappa(u)$, and suppose that the hypothesis of Theorem 3.3 be satisfied. Then any homogeneous blowup u_0 as in Lemma 6.9 is nonzero.*

Proof. If u_0 were zero, then there would exist j_0 such that $\sup_{\overline{B}_1} |u_{r_j}| \le \frac{c}{2}$ for all $j > j_0$, where c is the constant in Lemma 10.2. Therefore,

$$\sup_{\overline{B}_{r_j}} |u| \le \frac{c r_j^\kappa}{2}.$$

However, by Lemma 10.2 we have $c r_j^\kappa \le \sup_{\overline{B}_{r_j}} |u|$, which gives a contradiction. □

Remark 10.4. The arguments of Lemmas 6.9, 8.5 and 10.3 and Proposition 6.12 immediately show that if $u_r \to u_0$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$ for $r = r_j \to 0+$, then $u_0 \in \mathfrak{P}_\kappa^+(\mathbb{R}^n)$.

We are now ready to prove the uniqueness of the homogeneous blowups.

Theorem 10.5. *(Uniqueness of the homogeneous blowup at singular points) Let $u \in \mathfrak{S}$ and assume that $0 \in \Sigma_\kappa(u)$. Suppose that the hypothesis of Theorem 3.3 be satisfied. Then there exists a unique $p_\kappa \in \mathfrak{P}_\kappa^+(\mathbb{R}^n)$ such that the homogeneous scalings u_r converge in $C^{1,\alpha}_{loc}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$ to p_κ.*

Proof. Lemmas 6.9, 8.5 and 10.3 guarantee the existence of such a polynomial, so we are left with proving uniqueness. Let $u_{r_j} \to u_0$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n_\pm \cup \mathbb{R}^{n-1})$ for a certain sequence $r_j \to 0^+$. By

Remark 10.4 we know that $u_0 \in \mathfrak{P}^+_\kappa(\mathbb{R}^n)$. We then apply Corollary 9.4 to u and u_0. Since the limit $M_\kappa(u,u_0,0^+)$ exists, it can be computed as

$$M_\kappa(u,u_0,0^+) = \lim_{r_j\to 0^+} M_\kappa(u,u_0,r_j) = \lim_{r_j\to 0+} \int_{S_1} (u_{r_j}-u_0)^2\mu(r_jy) = 0,$$

where the second to the last equality holds because, thanks to the homogeneity of u_0, we have

$$\begin{aligned} M_\kappa(u,u_0,r) &= \frac{1}{r^{n-1+2\kappa}}\int_{S_r}(u-u_0)^2\mu = \frac{1}{r^{2\kappa}}\int_{S_1}(u(ry)-u_0(ry))^2\mu(ry) \\ &= \int_{S_1}(u_r-u_0)^2\mu(ry). \end{aligned}$$

Now, since $M_\kappa(u,u_0,0^+)=0$, we obtain for any $r\to 0^+$, and not just for the above sequence $r_j \to 0^+$, that

$$M_\kappa(u,u_0,r) = \int_{S_1}(u_r-u_0)^2\mu(ry) \ \to 0.$$

Therefore, if u_0' is a limit of u_r over another sequence $r=r_j'\to 0$, then

$$\int_{S_1}(u_0-u_0')^2 \le 2\lambda^{-1}\int_{S_1}(u_0-u_r)^2\mu(ry) + 2\lambda^{-1}\int_{S_1}(u_r-u_0')^2\mu(ry)\to 0.$$

This implies that $u_0 = u_0'$ in S_1. Since both u_0 and u_0' are homogeneous of degree κ, they must coincide in $\mathbb{R}^n$. □

Lemma 10.6. *Let $u \in \mathfrak{S}$. Then the set Σ_κ is of type F_σ, i.e., it is a union of countably many closed sets.*

Proof. The proof follows that of Lemma 1.5.3 in [GP]. Let

$$E_j = \Big\{x_0 \in \Sigma_\kappa(u)\cap \overline{B_{1-1/j}} \mid \rho^\kappa/j \le \sup_{S_\rho}|u_{x_0}(x)| \le j\rho^\kappa, \text{ for } 0<\rho<\lambda^{1/2}(1-|x_0|)\Big\}.$$

By Lemmas 4.2 and 10.2, $\Sigma_\kappa(u) = \cup_{j=1}^\infty E_j$, so we only need to prove that E_j is closed. Let $x_0 \in \overline{E_j}$. Then clearly $x_0 \in \overline{B_{1-1/j}}$ and

$$\frac{1}{j}\rho^\kappa \le \sup_{S_\rho}|u_{x_0}(x)| \le j\rho^\kappa, \quad \text{for } 0<\rho<\lambda^{1/2}(1-|x_0|), \tag{10.9}$$

therefore it suffices to prove that $x_0 \in \Sigma_\kappa(u)$. By Theorem 8.4, since κ is even, it suffices to show that $\tilde N_{L_{x_0}}(u,0^+)=\kappa$. By the upper semicontinuity of the function $x\mapsto \tilde N_{L_x}(u,0^+)$, we have that $\tilde N_{L_{x_0}}(u,0^+)\ge\kappa$. If we had $\tilde N_{L_{x_0}}(u,0^+)=\ell>\kappa$, then by Lemma 4.2 we would have $|u_{x_0}(x)|\le C_1|x|^\ell$ in a small enough ball, contradicting (10.9). Therefore, $\tilde N_{L_{x_0}}(0+,u)=\kappa$, proving the result. □

Theorem 10.7 (Continuous dependence of the blowups)**.** *Let $u\in\mathfrak{S}$. Given $x_0\in\Sigma_\kappa(u)$, with $\kappa>\frac{3}{2}$, denote by $p_\kappa^{x_0}$ the homogeneous blowup of u at x_0 as in Theorem 10.5, so that*

$$u(x) = p_\kappa^{x_0}(A^{-1/2}(x_0)(x-x_0)) + o(|A^{-1/2}(x_0)(x-x_0)|^\kappa).$$

Then the mapping $x_0\to p_\kappa^{x_0}$ from $\Sigma_\kappa(u)$ to $\mathfrak{P}^+_\kappa(\mathbb{R}^n)$ is continuous, where $\mathfrak{P}^+_\kappa(\mathbb{R}^n)$ is like in Definition 8.3. Moreover, for any compact $K\subset\Sigma_\kappa(u)\cap B_1$, there exists a modulus of continuity σ_K, with $\sigma_K(0^+)=0$, such that

$$|u(x)-p_\kappa^{x_0}(A^{-1/2}(x_0)(x-x_0))| \le \sigma_K(|A^{-1/2}(x_0)(x-x_0)|)|A^{-1/2}(x_0)(x-x_0)|^\kappa, \tag{10.10}$$

for any $x_0 \in K$.

Proof. The proof follows the ideas of Theorem 1.5.5 in [GP] and is included for clarity and completeness. $\mathfrak{P}_\kappa^+(\mathbb{R}^n)$ is a convex subset of the finite-dimensional vector space of all κ-homogeneous polynomials, therefore all norms are equivalent. We will endow such space with the norm of $L^2(S_1)$.

Given $x_0 \in \Sigma_\kappa(u)$ and $\varepsilon > 0$ small enough, there exists $r_\varepsilon = r_\varepsilon(x_0) > 0$ such that

$$M_\kappa^{x_0}(u, p_\kappa^{x_0}, r_\varepsilon) := M_\kappa(u_{x_0}, p_\kappa^{x_0}, r_\varepsilon) = \frac{1}{r_\varepsilon^{n-1+2\kappa}} \int_{S_{r_\varepsilon}} (u_{x_0} - p_\kappa^{x_0})^2 \mu < \varepsilon,$$

where we recall that $u_{x_0}(x) = u(x_0 + A^{1/2}(x_0)x)$. This implies that there exists $\delta_\varepsilon = \delta_\varepsilon(x_0) > 0$ such that if $z_0 \in \Sigma_\kappa(u) \cap B_{\delta_\varepsilon}(x_0)$, then

$$M_\kappa^{z_0}(u, p_\kappa^{x_0}, r_\varepsilon) = \frac{1}{r_\varepsilon^{n-1+2\kappa}} \int_{S_{r_\varepsilon}} (u_{z_0} - p_\kappa^{x_0})^2 \mu < 2\varepsilon.$$

Since $M_\kappa^{z_0}(u, p_\kappa^{x_0}, \cdot) + C^\star r$ is monotone nondecreasing, we conclude that for r_ε small enough

$$M_\kappa^{z_0}(u, p_\kappa^{x_0}, r) < 3\varepsilon, \ 0 < r < r_\varepsilon.$$

Letting $r \to 0^+$, we obtain

$$M_\kappa^{z_0}(u, p_\kappa^{x_0}, 0^+) = \int_{S_1} (p_\kappa^{z_0} - p_\kappa^{x_0})^2 \leq 3\varepsilon,$$

which concludes the first part of the theorem. To prove the second part, notice that for $|z_0 - x_0| < \delta_\varepsilon$ and $0 < r < r_\varepsilon$,

$$\begin{aligned} \|u_{z_0} - p_\kappa^{z_0}\|_{L^2(S_r)} &\leq \|u_{z_0} - p_\kappa^{x_0}\|_{L^2(S_r)} + \|p_\kappa^{x_0} - p_\kappa^{z_0}\|_{L^2(S_r)} \\ &\leq 2(3\varepsilon)^{1/2} r^{\frac{n-1}{2}+\kappa} \lambda^{-1/2}. \end{aligned}$$

Integrating in r, this also gives an estimate for solid integrals

$$\|u_{z_0} - p_\kappa^{z_0}\|_{L^2(B_{2r})} \leq C\varepsilon^{1/2} r^{\frac{n}{2}+\kappa}.$$

To proceed, we now notice that

$$L_{z_0} p_\kappa^{z_0} = L_{z_0} p_\kappa^{z_0} - \Delta p_\kappa^{z_0} = \operatorname{div}((A_{z_0} - I)\nabla p_\kappa^{z_0}) = \nabla A_{z_0} \nabla p_\kappa^{z_0} + (A_{z_0} - I) D^2 p_\kappa^{z_0}$$

and hence

$$|L_{z_0} p_\kappa^{z_0}| \leq C r^{\kappa-1} \quad \text{in } B_{2r}.$$

This then implies (by using the Signorini boundary conditions)

$$|L_{z_0}(u_{z_0} - p_\kappa^{z_0})^\pm| \leq C r^{\kappa-1} \quad \text{in } B_{2r},$$

and consequently that

$$\begin{aligned} \|u_{z_0} - p_\kappa^{z_0}\|_{L^\infty(B_r)} &\leq C r^{-n/2} \|u_{z_0} - p_\kappa^{z_0}\|_{L^2(B_{2r})} + C r^{\kappa+1} \\ &\leq C\varepsilon^{1/2} r^\kappa + C r^{\kappa+1}, \end{aligned}$$

by the interior L^∞-L^2 estimates. Rescaling, this gives

$$\|u_{z_0,r} - p_\kappa^{z_0}\|_{L^\infty(B_1)} \leq C(\varepsilon^{1/2} + r) \leq C_\varepsilon, \tag{10.11}$$

for $r < r_\varepsilon$ small, and $C_\varepsilon \to 0$ as $\varepsilon \to 0$, where we recall that $u_{z_0,r}(x) := u_{z_0}(rx)/r^\kappa$. Let now $K \subset \Sigma_\kappa(u) \cap B_1$ be compact. After covering it with finitely many balls $B_{\delta_\varepsilon(x_0^i)}(x_0^i)$ for some

$x_0^i \in K$, $i = 1, \dots, N$, we conclude that (10.11) holds for all $z_0 \in K$ if $r < r_\varepsilon^K := \min\{r_\varepsilon(x_0^i) \mid i = 1, \dots, N\}$. □

11. Structure of the singular set

We are now ready to prove our main result. We need the following definition.

Definition 11.1 (Dimension at the singular point)**.** Given a singular point $x_0 \in \Sigma_\kappa(u)$ we define the *dimension* of $\Sigma_\kappa(u)$ at x_0 to be

$$d_\kappa^{x_0} := \dim \Big\{ \xi \in \mathbb{R}^{n-1} \mid \langle \xi, \nabla_{x'} p_\kappa^{x_0}(x', 0) \rangle = 0,\ \forall x' \in \mathbb{R}^{n-1} \Big\}.$$

Notice that, since $p_\kappa^{x_0} \not\equiv 0$ on $\mathbb{R}^{n-1} \times \{0\}$ (by the Cauchy-Kovalevskaya theorem), we have $0 \leq d_\kappa^{x_0} \leq n-2$. Therefore, given $d \in \{0, \dots, n-2\}$, we define

$$\Sigma_\kappa^d(u) := \{x_0 \in \Sigma_\kappa(u) \mid d_\kappa^{x_0} = d\}.$$

Theorem 11.2 (Structure of the singular set)**.** *Let $u \in \mathfrak{S}$. Then $\Gamma_\kappa(u) = \Sigma_\kappa(u)$ for $k = 2m$, $m \in \mathbb{N}$. Moreover, every set $\Sigma_\kappa^d(u)$, $d \in \{0, \dots, n-2\}$, is contained in a countable union of d-dimensional C^1 manifolds.*

Proof. The claim that $\Gamma_\kappa(u) = \Sigma_\kappa(u)$ for $k = 2m$, $m \in \mathbb{N}$, was proved in Theorem 8.4. The proof of the structure of $\Sigma_\kappa^d(u)$ is based on Whitney's extension theorem, see [W], and the implicit function theorem, and follows the proof of Theorem 1.3.8 in [GP] with appropriate modifications. We include it here for completeness.

Step 1: Whitney's extension. Recall the definition of the sets E_j introduced in Lemma 10.6:

$$K = E_j = \{x_0 \in \Sigma_\kappa(u) \cap \overline{B_{1-1/j}} \mid \frac{1}{j}\rho^\kappa \leq \sup_{S_\rho} |u_{x_0}(x)| \leq j\rho^\kappa \text{ for } 0 < \rho < \lambda^{1/2}(1 - |x_0|)\}.$$

We have already proved that $\Gamma_\kappa(u) = \bigcup_{j=1}^\infty E_j$, where each E_j is compact.

If $p_\kappa^{x_0}$ denotes the unique homogeneous blowup of u at x^0, write

$$p_\kappa^{x_0}(x) = \sum_{|\alpha| = \kappa} \frac{a_\alpha(x)}{\alpha!} x^\alpha.$$

By Theorem 10.7, the coefficients a_α are continuous on $\Sigma_\kappa(u)$. Furthermore, combining (10.10) with the fact that $u(x) = 0$ on $\Sigma_\kappa(u)$, we obtain

$$|p_\kappa^{x_0}(A^{-1/2}(x_0)(x - x_0))| \leq \sigma(|A^{-1/2}(x_0)(x - x_0)|)|A^{-1/2}(x_0)(x - x_0)|^\kappa, \quad \text{for } x, x_0 \in K,$$

where $\sigma = \sigma_K$. For any multi-index α with $|\alpha| \leq \kappa$, define, for $x \in \Sigma_\kappa(u)$,

$$f_\alpha(x) = \begin{cases} 0, & |\alpha| \leq \kappa \\ a_\alpha(x), & |\alpha| = \kappa. \end{cases}$$

We will prove now a compatibility condition which will allow us to apply Whitney's extension theorem.

Lemma 11.3. *For any $x_0, x \in K$,*

$$f_\alpha(x) = \sum_{|\beta| \leq \kappa - |\alpha|} \frac{f_{\alpha+\beta}(x_0)}{\beta!} (x - x_0)^\beta + R_\alpha(x, x_0), \tag{11.1}$$

where for some modulus of continuity $\sigma_\alpha = \sigma_\alpha^K$,

$$|R_\alpha(x,x_0)| \le \sigma_\alpha(|x-x_0|)|x-x_0|^{\kappa-|\alpha|}. \tag{11.2}$$

Proof. (1) Consider the case $|\alpha| = \kappa$. Then $f_\alpha(x) = a_\alpha(x)$ and we need to prove that $R_\alpha(x,x_0) := a_\alpha(x) - a_\alpha(x_0)$ is such that $|R_\alpha(x,x_0)| \le \sigma_\alpha(|x-x_0|)$. Since $x \mapsto a_\alpha(x)$ is continuous on K, this claim holds.

(2) For $0 \le |\alpha| < \kappa$, we have $f_\alpha(x) = 0$, while the right-hand side of (11.1) is

$$\sum_{|\alpha+\beta|=\kappa} \frac{f_{\alpha+\beta}(x_0)}{\beta!}(x-x_0)^\beta + R_\alpha(x,x_0) = \sum_{|\gamma|=\kappa,\gamma>\alpha} \frac{a_\gamma(x_0)}{(\gamma-\alpha)!}(x-x_0)^{\gamma-\alpha} + R_\alpha(x,x_0)$$

and it suffices to prove that

$$R_\alpha(x,x_0) := -\sum_{|\gamma|=\kappa,\gamma>\alpha} \frac{a_\gamma(x_0)}{(\gamma-\alpha)!}(x-x_0)^{\gamma-\alpha} = -\partial^\alpha p_\kappa^{x_0}(x-x_0)$$

satisfies (11.2). Assume by contradiction that there exists no modulus of continuity σ_α such that (11.2) is satisfied for every $x_0, x \in K$. Then there exists $\delta > 0$ and sequences $x_0^i, x^i \in K$ such that $|x_0^i - x^i| := \rho_i \to 0$ and

$$\left| \sum_{|\gamma|=\kappa,\gamma>\alpha} \frac{a_\gamma(x_0^i)}{(\gamma-\alpha)!}(x^i-x_0^i)^{\gamma-\alpha} \right| \ge \delta |x^i - x_0^i|^{\kappa-|\alpha|}. \tag{11.3}$$

Consider the scalings

$$w^i(x) = \frac{u(x_0^i + \rho_i A^{1/2}(x_0^i)x)}{\rho_i^\kappa}, \quad \xi^i := (x^i - x_0^i)/\rho_i.$$

Without loss of generality, we may assume that $x_0^i \to x_0 \in K$ and $\xi^i \to \xi_0 \in S_1$. By Theorem 10.7,

$$\begin{aligned}\left|w^i(x) - p_\kappa^{x_0^i}(x)\right| &= \left|\frac{u(x_0^i + \rho_i A^{1/2}(x_0^i)x)}{\rho_i^\kappa} - p_\kappa^{x_0^i}(x)\right| = \frac{1}{\rho_i^\kappa}\left|u(x_0^i + \rho_i A^{1/2}(x_0^i)x) - p_\kappa^{x_0^i}(\rho_i x)\right| \\ &\le \frac{1}{\rho_i^\kappa}\sigma_K(|\rho_i x|)|\rho_i x|^\kappa = \sigma_K(|\rho_i x|)|x|^\kappa.\end{aligned}$$

This implies that w^i converges locally uniformly in $\mathbb{R}^n$ to $p_\kappa^{x_0}$, as

$$\begin{aligned}|w^i(x) - p_\kappa^{x_0}(x)| &\le |w^i(x) - p_\kappa^{x_0^i}(x)| + |p_\kappa^{x_0^i}(x) - p_\kappa^{x_0}(x)| \\ &\le \sigma_K(|\rho_i x|)|x|^\kappa + |p_\kappa^{x_0^i}(x) - p_\kappa^{x_0}(x)|,\end{aligned}$$

and the map $x_0 \mapsto p_\kappa^{x_0}$ from $\Sigma_\kappa(u)$ to $\mathcal{P}_\kappa$ is continuous.

Now notice that since $x^i \in K = E_j$, then

$$\frac{1}{j}\rho^\kappa \le \sup_{S_\rho} |u_{x^i}(x)| \le j\rho^\kappa, \quad \text{for } 0 < \rho < \lambda^{1/2}(1-|x^i|),$$

which can be rewritten as

$$\frac{1}{j}\rho^\kappa \le \sup_{|A^{-1/2}(x^i)(A^{1/2}(x_0^i)y-\xi^i)|=\rho} |w^i(y)| \le j\rho^\kappa, \quad \text{for } 0 < \rho\rho_i < \lambda^{1/2}(1-|x^i|).$$

Passing to the limit, we obtain

$$\frac{1}{j}\rho^\kappa \le \sup_{S_\rho(\xi_0)} |p_\kappa^{x_0}(x)| \le j\rho^\kappa, \quad 0<\rho<\infty.$$

This implies that $\xi_0 \in \Sigma_\kappa(p_\kappa^{x_0})$ and in particular that

$$\partial^\beta p_\kappa^{x_0}(\xi_0) = 0 \quad \text{for all } |\beta|<\kappa. \tag{11.4}$$

However, dividing both parts of (11.3) by $\rho_i^{\kappa-|\alpha|}$ and passing to the limit, we obtain

$$\partial^\alpha p_\kappa^{x_0}(\xi_0) = \sum_{\gamma=\kappa,\gamma>\alpha} \frac{a_\gamma(x_0)}{(\gamma-\alpha)!}\xi_0^{\gamma-\alpha} \ge \delta > 0,$$

contradicting (11.4) for $\beta=\alpha$. This completes the proof. □

The lemma above allows us to apply Whitney's extension theorem, concluding that there exists $F\in C^\kappa(\mathbb{R}^n)$ such that

$$\partial^\alpha F = f_\alpha, \quad \forall |\alpha|\le\kappa.$$

Step 2: Implicit function theorem. Let $x_0\in\Sigma_\kappa^d(u)\cap E_j$. By definition, this means that

$$d=\dim\{\xi\in\mathbb{R}^{n-1} \mid \langle \xi, \nabla_{x'} p_\kappa^{x_0}\rangle \equiv 0 \text{ on } \mathbb{R}^{n-1}\}.$$

We note that the equivalent definition of d is given by

$$d=\dim\{\xi\in\mathbb{R}^{n-1} \mid \langle \xi, \nabla_{x'}\partial_{x'}^{\beta'} p_\kappa^{x_0}\rangle = 0, \text{ for any } |\beta|=\kappa-1\}.$$

Therefore, there exist $n-1-d$ multi-indices β_i' of order $|\beta_i'|=\kappa-1$, $i=1,\dots,n-1-d$ such that

$$v_i = \nabla_{x'}\partial_{x'}^{\beta_i'} F(x_0) = \nabla_{x'}\partial_{x'}^{\beta_i'} p_\kappa^{x_0}$$

are linearly independent. On the other hand,

$$\Sigma_\kappa^d(u)\cap E_j \subset \bigcap_{i=1}^{n-1-d} \{\partial_{x'}^{\beta_i'}F=0\}.$$

Hence the implicit function theorem implies that $\Sigma_\kappa^d(u)\cap E_j$ is contained in a d-dimensional manifold in a neighborhood of x_0. Since $\Sigma_\kappa(u)=\bigcup_{j=1}^\infty E_j$, the theorem holds. □

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