

INFLECTION SETS OF PERMUTATIONS

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ABSTRACT. In 2020, Brualdi and Dahl introduced the discrete derivative $\sigma'(i) = \sigma(i+1) - \sigma(i)$ of a permutation $\sigma \in \text{Sym}_n$. In this paper, we focus on the second derivative of a permutation, $\sigma''(i) = \sigma(i+1) + \sigma(i-1) - 2\sigma(i)$, and the inflections it determines. We show that for even n , the only distinct permutations with the same second derivative are the identity and its reverse, while for odd n , the second derivative is not injective even after excluding them. Our results concern the characterization of sets of pairs of integers (i, j) that are the inflection set of some permutation and whether a given set is witnessed in Sym_n for infinitely many n . More specifically, we show that exactly 14 permutations of $[n]$ have no inflections when $n \geq 5$. For a single inflection $\{(k, m)\}$, we determine all n for which it is witnessed in Sym_n : we prove it is always witnessed, and it is witnessed for only finitely many n if and only if $m \geq 2k + 1$. In that case, all such n satisfy $2m - 2k + 1 \leq n \leq 2m - k + 2$ (apart from $k = 2, n = m + 1$), and, perhaps surprisingly, this range has gaps once $k \geq 10$, the smallest occurring at $(k, m, n) = (10, 21, 30)$. Finally, we characterize maximally inflected permutations, whose second derivative alternates in sign, and show that their number $|\mathcal{M}_n|$ satisfies $2E_n \leq |\mathcal{M}_n| \leq (2/3)^{\lfloor n/3 \rfloor - 1} n!$, where E_n is the Euler zigzag number.

1. INTRODUCTION

In this paper, we consider derivatives and inflections of permutations in Sym_n . Throughout, we let $n \in \mathbb{N} = \{1, 2, 3, \dots\}$, $[n] = \{1, 2, \dots, n\}$, $\pm[n] = [n] \cup \{-1, -2, \dots, -n\}$, and $[i, j] = \{i, i+1, \dots, j\}$ for integers $i \leq j$, with $[i, j] = \emptyset$ if $i > j$. We let Sym_n denote the set of permutations of $[n]$, and given a permutation $\pi \in \text{Sym}_n$ we write $\pi = \pi(1)\pi(2) \cdots \pi(n)$ in one-line notation. Our goal is to understand inflections of permutations. To do so, we define the first and second derivative of a permutation.

Definition 1.1. Let $\pi \in \text{Sym}_n$. We define the *derivative* of π to be the function $\pi' : [n-1] \rightarrow \mathbb{Z}$ given by $\pi'(j) = \pi(j+1) - \pi(j)$. We sometimes use D to denote the map $D : \text{Sym}_n \rightarrow \mathbb{Z}^{n-1}$ defined by $D(\pi) = \pi'$. Given an integer $2 \leq \ell \leq n-1$, we define the *second derivative of π at ℓ* to be

$$\pi''(\ell) := \pi(\ell+1) + \pi(\ell-1) - 2\pi(\ell).$$

With a notion of second derivatives at hand, we can define convexity/concavity.

Definition 1.2. We say $\pi \in \text{Sym}_n$ is *convex (concave) at $m \in [2, n-1]$* if $\pi(m+1) + \pi(m-1) - 2\pi(m) > 0$ (respectively, $\pi(m+1) + \pi(m-1) - 2\pi(m) < 0$).

Definition 1.3. Let $n \geq 4$, $\pi \in \text{Sym}_n$, and $m, p \in [2, n-1]$ with $m < p$. We say that π has an *inflection between m and p* if

- $\pi''(m)\pi''(p) < 0$ and
- $\pi''(\ell) = 0$ for all $m < \ell < p$.

We denote the inflection set of a permutation σ by $\mathcal{I}(\sigma)$, consisting of all pairs of integers (i, j) , $2 \leq i < j \leq n-1$ such that σ has an inflection between i and j .

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Example 1.4. Consider the permutation $\pi = 23468715 \in \text{Sym}_8$. We have

ℓ	2	3	4	5	6	7
$\pi''(\ell)$	0	1	0	-3	-5	10

This shows that π has inflections between 3 and 5 and between 6 and 7, which we denote as $\mathcal{I}(\pi) = \{(3, 5), (6, 7)\}$.

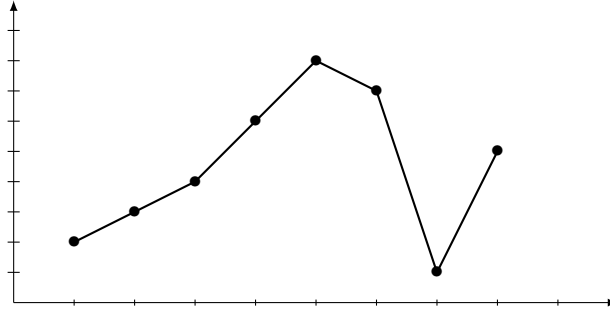


FIGURE 1. The graph of $\pi = 23468715$ with inflection set $\mathcal{I}(\pi) = \{(3, 5), (6, 7)\}$.

Definition 1.5. Fix $w \in \text{Sym}_n$. We call the *reverse* of w the permutation $w^r := w(n) \cdots w(1) \in \text{Sym}_n$, and the *complement* of w the permutation $\bar{w} \in \text{Sym}_n$ defined by $\bar{w}(i) = n + 1 - w(i)$. The *fundamental symmetries* of w are the (possibly non-distinct) permutations

$$\{w, w^r, \bar{w}, \bar{w}^r = \overline{w^r}\}.$$

The first derivative π' of a permutation was introduced by Brualdi and Dahl [BD20] under the name *discrete derivative*. They show that $D : \text{Sym}_n \rightarrow \mathbb{Z}^{n-1}$ is injective and characterize the integer vectors that arise as first derivatives. We recall these facts in Lemmas 2.1 and 2.2. They also classify *convex* permutations, which are those for which π' is non-decreasing. Since a permutation has no inflections exactly when it or its complement is convex, their classification is equivalent to Theorem 4.1, for which we give an independent proof. Using this result, we conclude that if $n \geq 5$, the number of permutations with no inflections is 14.

The second derivative π'' and the notion of inflection are not considered in [BD20]. In particular, the question of when two permutations share a second derivative, and the study of permutations with a prescribed inflection set, are new.

Our approach to inflection sets is modeled on the study of other position and value-based statistics. Billey, Burdzy and Sagan [BBS13] enumerate permutations with a given peak set, and Davis, Nelson, Petersen and Tenner [Dav+18] introduce the pinnacle set, characterize the admissible pinnacle sets, and study how many permutations realize each one. Analogous questions have been studied by the authors, together with González, Harris, and Tenner, in the context of signed permutations of types B and D [Gon+23] and for Stirling permutations, where pinnacles are replaced by mesas [Gon+25]. See also [Dia+21]. We ask the same two questions for inflections: which sets occur as the inflection set of some $\sigma \in \text{Sym}_n$, and how many witnesses each has. A feature with no counterpart for peaks or pinnacles is that admissibility of an inflection set depends on n ; a set may be realized in Sym_n for some values of n and for no others.

Maximally inflected permutations generalize alternating permutations (see [Sta10] for a survey), which are exactly the permutations whose first derivative alternates in sign. Here, we ask instead that the second derivative alternates in sign. In Theorem 6.10, we obtain a necessary condition on run lengths, which places maximally inflected permutations within the family of permutations with restricted run structure, whose enumeration is studied in [FS14; Zhu16]. Our bounds on the number of maximally inflected permutations of Sym_n are in the spirit of the asymptotics of

consecutive pattern avoidance [EN03; EKP11], where the number of permutations in Sym_n avoiding a fixed consecutive pattern grows like $c\lambda^n n!$ for constants $c > 0$ and $0 < \lambda < 1$. Asymptotics of the same shape, obtained through the spectrum of an integral operator, also govern permutations with a periodic pattern of rises and falls [BHR03]. Omar and Troyka [OT26] study growth rates of permutations with a fixed descent or peak set. However, maximal inflection is not a condition on relative order alone, since the sign of $\sigma''(i)$ depends on the values $\sigma(i-1), \sigma(i), \sigma(i+1)$, so none of these results applies directly.

Finally, $\sigma''(i) = 0$ exactly when $\sigma(i-1), \sigma(i), \sigma(i+1)$ are in an arithmetic progression. Permutations with σ'' nowhere zero are thus those avoiding three consecutive terms in arithmetic progression, counted by [OEI26b]. This is different from the classical problem of avoiding three-term arithmetic progressions at arbitrary positions [Dav+77].

The outline of the paper is as follows. In Section 2, we recall results from [BD20] pertaining to the injectivity of the first derivative. Our first main result analyzes the injectivity of the second derivative in Sym_n for n even and odd. When n is even, we show that the only two distinct permutations with the same second derivative are the identity (denoted by e) and its reverse e^r . When n is odd, we provide examples of permutations, distinct from e and e^r , with the same second derivative. In Section 3, we define admissible inflection sets. Those are divided into two types, based on whether there are infinitely many n such that there is a witness in Sym_n .

In Section 4, we provide a different proof of [BD20, Theorem 6], which characterizes permutations with no inflections. We also prove that the number of permutations of $[n]$ with no inflections is eventually constant for all sufficiently large n .

In Section 5, we show that any inflection set of size one is admissible, that is, given $2 \leq k < m$, $I = \{(k, m)\}$ is an admissible inflection set. Moreover, our second and third main results, Theorems 5.10 and 5.14, determine, in terms of k , m , and n , exactly when I has a witness in Sym_n . As a consequence, we show that given $2 \leq k < m$ and $I = \{(k, m)\}$, I is vanishing if and only if $m \geq 2k + 1$.

In Section 6, we study the size of $\mathcal{M}_n$, the set of maximally inflected permutations in Sym_n . We first characterize maximally inflected permutations (Theorem 6.4) and show that alternating permutations are maximally inflected (Corollary 6.5). Using classical asymptotics of the Euler zigzag numbers, this gives $\lim_{n \rightarrow \infty} |\mathcal{M}_n|^{1/n} = \infty$. The rest of the section addresses how $|\mathcal{M}_n|$ compares to $n!$. We give two upper bounds on the fraction $|\mathcal{M}_n|/n!$, both probabilistic. We first prove a necessary condition for a permutation to be maximally inflected: every window of k consecutive positions must standardize to a permutation whose interior runs all have odd length (Theorems 6.10 and 6.13). Via Fekete's Lemma, we are able to bound $\limsup_n (|\mathcal{M}_n|/n!)^{1/n}$ by $\lim_n p_n^{1/n}$, where p_n is the proportion of permutations in Sym_n all of whose interior runs have odd length. Our last main result, Theorem 6.15, samples a uniformly random σ and shows that each block of three consecutive values, conditioned on the past, is consistent with maximal inflection with probability at most $2/3$. Iterating over the $\lfloor n/3 \rfloor$ blocks gives the explicit bound $|\mathcal{M}_n|/n! \leq (2/3)^{\lfloor n/3 \rfloor - 1}$, hence $|\mathcal{M}_n|/n! \rightarrow 0$ with $\limsup_n (|\mathcal{M}_n|/n!)^{1/n} \leq (2/3)^{1/3}$.

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2. INJECTIVITY OF DERIVATIVES

In this section, we discuss the injectivity of the first and second derivatives in Sym_n .

2.1. Injectivity of the first derivative. The injectivity of the first derivative has been settled.

Lemma 2.1 ([BD20, Proposition 2]). *The map $D : \text{Sym}_n \rightarrow \mathbb{Z}^{n-1}$ defined $D(\pi) = \pi'$ is injective.*

The following result characterizes which $(n-1)$ -tuples in $(\pm[n-1])^{n-1}$ are derivatives of a permutation.

Lemma 2.2 ([BD20, Proposition 4]). *Given $\beta \in (\pm[n-1])^{n-1}$, there exists $\alpha \in \text{Sym}_n$ such that $\alpha' = \beta$ if and only if $0 < |\sum_{i=j}^k \beta_i| \leq n-1$ for all $j, k \in [n-1]$ with $j \leq k$.*

Example 2.3. Lemma 2.2 yields the following 12 first derivatives of permutations $\pi \in \text{Sym}_4$ with $\pi'(1) > 0$ (the other 12 are their negatives). We use $\bar{i}$ to denote $-i$.

$$111 \quad 3\bar{1}\bar{1} \quad 3\bar{2}\bar{1} \quad 2\bar{1}\bar{2} \quad 2\bar{1}\bar{2} \quad 2\bar{1}\bar{2} \quad 2\bar{3}\bar{2} \quad 12\bar{1} \quad 11\bar{3} \quad 1\bar{2}\bar{3} \quad 1\bar{2}\bar{1} \quad 1\bar{3}\bar{1}$$

2.2. (Lack of) Injectivity of the second derivative.

Lemma 2.4. *Let $\sigma, \pi \in \text{Sym}_{2k+1}$ for some nonnegative integer k . If $\sigma'' = \pi''$, then $\sigma(k+1) = \pi(k+1)$.*

Proof. Notice that

$$\begin{aligned} \binom{2}{2}\sigma''(2) + \cdots + \binom{k}{2}\sigma''(k) + \binom{k+1}{2}\sigma''(k+1) + \binom{k}{2}\sigma''(k+2) + \cdots + \binom{2}{2}\sigma''(2k) \\ = \sigma(1) + \cdots + \sigma(k) - 2k\sigma(k+1) + \sigma(k+2) + \cdots + \sigma(2k+1). \end{aligned}$$

Since $\sigma, \pi \in \text{Sym}_{2k+1}$,

$$\sigma(1) + \cdots + \sigma(2k+1) = \frac{(2k+1)(2k+2)}{2} = \pi(1) + \cdots + \pi(2k+1).$$

Therefore if $\sigma'' = \pi''$, then

$$\frac{(2k+1)(2k+2)}{2} - (2k+1)\sigma(k+1) = \frac{(2k+1)(2k+2)}{2} - (2k+1)\pi(k+1).$$

Consequently, $\sigma(k+1) = \pi(k+1)$. □

More generally, we can extract similar information by looking at a weighted sum of second derivatives:

Lemma 2.5. *Let $\sigma, \pi \in \text{Sym}_n$. If $\sigma'' = \pi''$, then for each $1 \leq k \leq n-1$,*

$$A(k)\sigma(k) + B(k)\sigma(k+1) = A(k)\pi(k) + B(k)\pi(k+1),$$

where $A(k) = \frac{n(n-1-2k)}{2}$ and $B(k) = -\frac{n(n+1-2k)}{2}$.

Proof. Given $1 \leq k \leq n-1$, we compute directly

$$\sum_{i=2}^k \binom{i}{2}\sigma''(i) + \sum_{i=k+1}^{n-1} \binom{n-i+1}{2}\sigma''(i) = A(k)\sigma(k) + B(k)\sigma(k+1) + \sum_{i=1}^n \sigma(i),$$

where

$$A(k) = \binom{k-1}{2} - 2\binom{k}{2} + \binom{n-k}{2} - 1 = \frac{n(n-1-2k)}{2},$$

and

$$B(k) = \binom{k}{2} - 2\binom{n-k}{2} + \binom{n-k-1}{2} - 1 = -\frac{n(n+1-2k)}{2}.$$

Since

$$\sum_{i=1}^n \sigma(i) = \sum_{i=1}^n \pi(i) = \frac{(n+1)n}{2},$$

this yields

$$A(k)\sigma(k) + B(k)\sigma(k+1) = A(k)\pi(k) + B(k)\pi(k+1)$$

for $1 \leq k \leq n-1$. □

Remark 2.6. We notice that

$$\gcd(A(k), B(k)) = \begin{cases} n/2, & \text{if } n \text{ is even} \\ n, & \text{if } n \text{ is odd.} \end{cases}$$

Indeed, we have $A(k) = \frac{n(n-2k-1)}{2}$ and $B(k) = -\frac{n(n-2k+1)}{2}$. Since $\gcd(a, b) = \gcd(a, b-a)$, $\gcd\left(\frac{n(n-2k-1)}{2}, \frac{n(n-2k+1)}{2}\right) = \gcd\left(\frac{n(n-2k-1)}{2}, n\right)$. If n is odd, then $n-2k-1$ is even, and n divides $A(k)$, giving $\gcd(A(k), B(k)) = n$.

If n is even, then $n-2k-1$ is odd. Writing $n = 2\ell$, we have $\frac{n(n-2k-1)}{2} = \ell(n-2k-1)$ and $\gcd(A(k), B(k)) = \gcd(\ell(n-2k-1), 2\ell) = \ell \gcd(n-2k-1, 2) = \ell = n/2$.

Example 2.7. Let $n = 7$ and let $\sigma, \pi \in \text{Sym}_7$ with $\sigma'' = \pi''$. Here $A(k) = 7(3-k)$ and $B(k) = -7(4-k)$, so

k	1	2	3	4	5	6
$A(k)$	14	7	0	-7	-14	-21
$B(k)$	-21	-14	-7	0	7	14

Dividing each equation of Lemma 2.5 by $\gcd(A(k), B(k)) = 7$ gives

$$\begin{aligned} 2\sigma(1) - 3\sigma(2) &= 2\pi(1) - 3\pi(2) & (k=1) \\ \sigma(2) - 2\sigma(3) &= \pi(2) - 2\pi(3) & (k=2) \\ \sigma(4) &= \pi(4) & (k=3 \text{ and } k=4) \\ \sigma(6) - 2\sigma(5) &= \pi(6) - 2\pi(5) & (k=5) \\ 2\sigma(7) - 3\sigma(6) &= 2\pi(7) - 3\pi(6) & (k=6). \end{aligned}$$

Writing $\delta(i) = \sigma(i) - \pi(i)$, this system reads

$$2\delta(1) = 3\delta(2), \quad \delta(2) = 2\delta(3), \quad \delta(4) = 0, \quad \delta(6) = 2\delta(5), \quad 2\delta(7) = 3\delta(6).$$

The solutions are

$$(\delta(1), \dots, \delta(7)) = (3t, 2t, t, 0, s, 2s, 3s)$$

for integers s, t . Since $\sum_{i=1}^7 \delta(i) = 0$, we get $s = -t$, that is, $\delta(i) = t(4-i)$. Finally, $|\delta(1)| \leq 6$ forces $|t| \leq 2$. The case $t = 0$ corresponds to $\sigma = \pi$, and $t = \pm 2$ gives $\{\sigma, \pi\} = \{e, e^r\}$. The case $t = \pm 1$ occurs for other pairs of permutations: for instance, $\sigma = 4567123$ and $\pi = 1357246$ satisfy $\sigma'' = \pi'' = (0, 0, -7, 7, 0)$ and $\sigma - \pi = (3, 2, 1, 0, -1, -2, -3)$.

We start the analysis of the injectivity of the second derivative in Sym_n for n even by noticing that in that case, if two permutations have the same second derivative and coincide at one value, they must be the same:

Lemma 2.8. *Let $\sigma, \pi \in \text{Sym}_n$ be such that $\sigma'' = \pi''$. If n even and there exists $i \in [n]$ with $\sigma(i) = \pi(i)$, then $\sigma = \pi$.*

Proof. By Lemma 2.5, for every $1 \leq k \leq n-1$,

$$A(k)(\sigma(k) - \pi(k)) + B(k)(\sigma(k+1) - \pi(k+1)) = 0,$$

where $A(k) = \frac{n(n-1-2k)}{2}$ and $B(k) = -\frac{n(n+1-2k)}{2}$. Also notice that $A(k) = 0$ if and only if $k = \frac{n-1}{2}$, and $B(k) = 0$ if and only if $k = \frac{n+1}{2}$, neither of which happens when n is even. Consequently, if n is even, we can write for $1 \leq k \leq n-1$,

$$\sigma(k+1) - \pi(k+1) = -\frac{A(k)}{B(k)}(\sigma(k) - \pi(k)).$$

Therefore $\sigma(j) - \pi(j) = 0$ if and only if $\sigma(j+1) - \pi(j+1) = 0$ for every $1 \leq j \leq n-1$. This gives a chain of equalities showing that if $\sigma(i) = \pi(i)$ for some $i \in [n]$, then $\sigma(j) = \pi(j)$ for all $j \in [n]$. $\square$

The following theorem proves when n is even, the only two distinct permutations with the same second derivative are the identity and its reverse:

Theorem 2.9. *Let σ, π be distinct permutations of Sym_n , where n is even. If $\sigma'' = \pi''$, then $\sigma = \pi^r$ and σ or π is the identity.*

Proof. By Lemma 2.5 we know that $(n-3)\sigma(1) - (n-1)\sigma(2) = (n-3)\pi(1) - (n-1)\pi(2)$. Rearranging terms yields

$$\frac{n-3}{n-1}(\sigma(1) - \pi(1)) = \sigma(2) - \pi(2)$$

Since n is even, $\gcd(n-3, n-1) = 1$. Hence, $(n-1) \mid (\sigma(1) - \pi(1))$, where $-n < \sigma(1) - \pi(1) < n$. If $\sigma(1) - \pi(1) = 0$, then by Lemma 2.8 we would obtain $\sigma = \pi$, contradicting the assumption that they are distinct. Hence, $|\sigma(1) - \pi(1)| = n-1$ so that $\sigma(1) = 1, \pi(1) = n$ or $\sigma(1) = n, \pi(1) = 1$. Assume without loss of generality that $\sigma(1) = 1$ and $\pi(1) = n$.

We now use strong induction to finish the proof. Let $2 \leq i \leq n$ and assume $\sigma(j) = j$ and $\pi(j) = n+1-j$ for all $1 \leq j < i$. Then $\sigma(i) \geq i$ and $\pi(i) \leq n-i+1$. Therefore

$$\sigma(i) - \pi(i) \geq i - (n-i+1) = 2i - n - 1, \quad (1)$$

with equality forcing $\sigma(i) = i$ and $\pi(i) = n-i+1$. Lemma 2.5 (with $k = i-1$) gives

$$\frac{n(n-1-2(i-1))}{2}(i-1 - (n+1-(i-1))) = -\frac{n(n+1-2(i-1))}{2}(\pi(i) - \sigma(i))$$

that is,

$$\frac{n(n+1-2i)}{2}(2i-n-3) = -\frac{n(n+3-2i)}{2}(\pi(i) - \sigma(i))$$

hence $(n+1-2i) = \pi(i) - \sigma(i)$, from which (1) gives us $\sigma(i) = i$ and $\pi(i) = n-i+1$, as desired. In this case, $\sigma = e$ and $\pi = e^r$. $\square$

When n is odd, the second derivative is not injective, even when excluding the identity and its reverse:

Theorem 2.10. *Let $k \geq 1$. The second derivative is not injective on $\text{Sym}_{2k+1} \setminus \{e, e^r\}$.*

Proof. Let $n = 2k+1$, and take $\sigma = 135 \cdots n24 \cdots (n-1)$ and $\pi = (k+1)(k+2) \cdots n123 \cdots k$. Then $\sigma'' = 0 \cdots 0(-n)n0 \cdots 0 = \pi''$. $\square$

Remark 2.11. For odd $n = 2m+1$, two distinct permutations $\sigma, \pi \in \text{Sym}_n$ with $\sigma'' = \pi''$ satisfy $\sigma(i) - \pi(i) = c(i - (m+1))$ for some $c \in \{\pm 1, \pm 2\}$, and $c = \pm 2$ only for $\{e, e^r\}$. The pairs with $c = \pm 1$ correspond to permutations γ of $\{-m, \dots, m\}$ for which $j \mapsto \gamma(j) + j$ is a permutation. These are the additive permutations of Kotzig and Laufer [KL78], counted by [OEI26a]. Hence the number of unordered pairs of distinct permutations in Sym_{2k+1} with the same second derivative is $1 + a(k)$, where $a(k)$ denotes the k th term of [OEI26a] (so $a(1), \dots, a(4) = 2, 6, 28, 244$). For $n = 3, 5, 7, 9$ this gives 3, 7, 29, 245 pairs, respectively.

3. ADMISSIBLE INFLECTION SETS

We say that if a set of pairs of integers I is such that there exists a permutation σ with inflection set $\mathcal{I}(\sigma) = I$, the set is *witnessed* by it. The sets that are witnessed are split into two types, based on whether there are infinitely many n such that there is a witness in Sym_n . In this section, we provide examples to illustrate inadmissibility and each of the two types of admissibility.

Definition 3.1. We say that a set of pairs of integers I is *admissible* if there exist n and a permutation $w \in \text{Sym}_n$ (called a *witness* for I) such that $\mathcal{I}(w) = I$. We say that an admissible inflection set I is *non-vanishing* if there exist infinitely many n so that there is a witness in Sym_n . Otherwise, we say that I is *vanishing*.

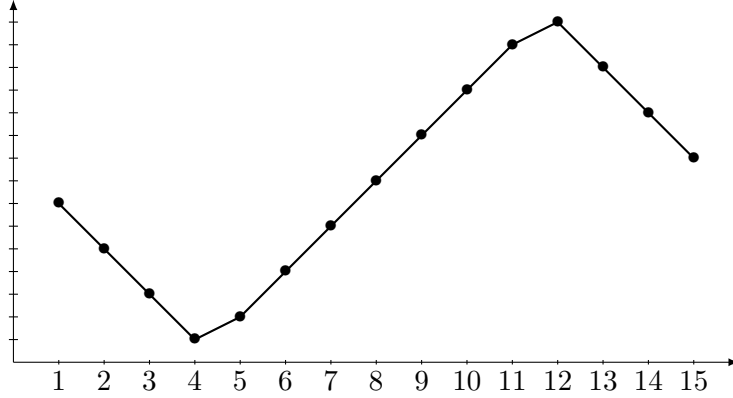


FIGURE 2. The graph of a permutation witnessing inflection set $\{(5, 11)\}$.

Example 3.2 (Admissible vanishing). Consider the set $\{(5, 11)\}$. The permutation

$$\sigma = 7 \ 5 \ 3 \ 1 \ 2 \ 4 \ 6 \ 8 \ 10 \ 12 \ 14 \ 15 \ 13 \ 11 \ 9$$

illustrated in Figure 2 witnesses the inflection set $\{(5, 11)\}$. In fact, there is a witness for $\{(5, 11)\}$ in Sym_n if and only if $13 \leq n \leq 19$, see Theorem 5.10; in particular, $\{(5, 11)\}$ is vanishing.

Example 3.3 (Admissible non-vanishing). The set $\{(2, 3)\}$ is witnessed by $n1(n-1)(n-2)\dots 2 \in \text{Sym}_n$ for every $n \geq 4$. More generally, every set $\{(k, m)\}$ with $2 \leq k < m \leq 2k$ is non-vanishing, see Corollary 5.15.

However, it is not the case that every set of pairs of integers is admissible.

Example 3.4 (Inadmissible). The set $\{(3, 6), (6, 10)\}$ is not witnessed by any permutation. After relacing σ by its complement if necessary, we may assume that $\sigma''(3) < 0$, $\sigma''(6) > 0$, and $\sigma''(10) < 0$. Denote $\sigma'(3) = \sigma'(4) = \sigma'(5) = a$ and $\sigma'(6) = \sigma'(7) = \sigma'(8) = \sigma'(9) = b$, where $a < b$. Note that

$$\sigma'(1) \geq \sigma'(2) > a \quad \text{and} \quad b > \sigma'(10) \geq \sigma'(11) \geq \dots \geq \sigma'(n-1). \quad (2)$$

We refer to the consecutive subword $\sigma([11, n])$ as the suffix of σ , which must be concave. Furthermore, we denote the values below $\sigma(10)$ missed by the first 10 values of σ by C . That is,

$$C = [1, \sigma(10)] \setminus \{\sigma([1, 10])\} = \{c_1, c_2, \dots, c_k\},$$

where $c_1 < c_2 < \dots < c_k$. In particular, we must have that $\sigma(n-i+1) = c_i$ for $1 \leq i \leq k$, and $\sigma'(n-i) = c_i - c_{i+1}$ for $1 \leq i \leq k-1$. By (2), if there is some consecutive c_i, c_{i+1} then each of the values $c_{i+1}, \dots, c_k$, must be consecutive. Equivalently, if $c_{i-1} = c_i - 1$ for some $i < k$, but $[c_i, c_k] \setminus C$

is nonempty this contradicts (2) saying that the suffix σ is concave. We will use this condition to establish $a < 0 < b$ and then rule out all values of b .

First, assume that $a > 0$ to obtain a contradiction. Then $\sigma([1, 10])$ is increasing. If $\sigma'(1), \sigma'(2)$, or $b \geq 3$, then there are consecutive values $c_{i-1}, c_i \in C$ for some $i < k$, but $[c_i, c_k] \setminus C$ is nonempty. Hence, $\sigma'(1) = \sigma'(2) = b = 2$, and $a = 1$ so that

$$\sigma([1, 7]) = \sigma(1), \sigma(1) + 2, \sigma(1) + 4, \sigma(1) + 5, \sigma(1) + 6, \sigma(1) + 7, \sigma(1) + 9,$$

which implies that the $\sigma(1) + 8, \sigma(1) + 3, \sigma(1) + 1$ must be a consecutive subword of the suffix of σ , again contradicting (2). Therefore, $a < 0$.

Next, we will show that $b > 0$. Assume $b < 0$ to obtain a contradiction. Since $a < b < 0$ we have $a \leq -2$ and $b \leq -1$ so that $\sigma(3) - \sigma(10) = -3a - 4b \geq 10$. Also, $\sigma'(n-1) \leq \dots \leq \sigma'(10) < -1$ such that $1 \leq \sigma(n) < \sigma(n-1) < \dots < \sigma(10)$. Thus, all integers in the interval $[\sigma(10), \sigma(3)]$ must occur in the first 10 positions, but there are at least 11 integers in the interval $[\sigma(10), \sigma(3)]$, which is a contradiction. Therefore, we must have $a < 0 < b$.

Consider the consecutive subword

$$\sigma([3, 10]) = \sigma(6) - 3a, \sigma(6) - 2a, \sigma(6) - a, \sigma(6), \sigma(6) + b, \sigma(6) + 2b, \sigma(6) + 3b, \sigma(6) + 4b.$$

We split into the below cases and show each is impossible.

If $b = 1$, then $a \leq -5$ so that $\sigma(3) - \sigma(10) = -3a - 4 \geq 11$. However, the suffix of σ must be decreasing. Hence, all of the integers in the interval $[\sigma(10), \sigma(3)]$ of which there are least 12, must occur in the first 10 positions, which is a contradiction.

If $b = 2$, then $\sigma(4) = \sigma(6) - 2a \neq \sigma(7), \dots, \sigma(10)$ so that $a \leq -5$. Hence, $\sigma(4) > \sigma(10)$. Since $\sigma'(10) < 2$ the suffix of σ may have some portion of consecutive values, i.e. slope 1. However, it cannot reach any value larger than $\sigma(4)$. Therefore, the at least six integers in the interval $[\sigma(4), \sigma(3)]$ must be among $\sigma(1), \dots, \sigma(4)$, which is a contradiction.

If $b = 3$, then $\sigma(3) = \sigma(6) - 3a \neq \sigma(7), \dots, \sigma(10)$ so that $a \leq -5$. Concavity of the suffix of σ implies that two consecutive integers below $\sigma(9)$ cannot both be missing from the first 10 entries, $\sigma([1, 10])$. Thus each pair $\{\sigma(6) + 1, \sigma(6) + 2\}$, $\{\sigma(6) + 4, \sigma(6) + 5\}$, $\{\sigma(6) + 7, \sigma(6) + 8\}$ must contain at least one value occurring in $\sigma([1, 5])$. Since $\sigma(3), \sigma(4) \geq \sigma(6) + 10$, only $\sigma(1), \sigma(2)$, and $\sigma(5)$ can achieve these values. Hence, $\sigma(6) < \sigma(1), \sigma(2) < \sigma(6) + 9 = \sigma(9)$. Since $\sigma'(1) \geq \sigma'(2)$, $\sigma(2) - \sigma(1) \geq -3a + \sigma(6) - \sigma(2)$, i.e.

$$15 \leq -3a \leq 2\sigma(2) - \sigma(1) - \sigma(6) \leq 15.$$

But then $a = -5$, $\sigma(1) = \sigma(6) + 1$, and $\sigma(2) = \sigma(6) + 8$, which implies that $\sigma(6) + 7, \sigma(6) + 4, \sigma(6) + 2$ must be a consecutive subword of the suffix of σ , contradicting the concavity of the suffix.

If $b = 4$, then making sure $\{\sigma(6) - 3a, \sigma(6) - 2a, \sigma(6) - a\} \cap \{\sigma(6) + 4, \sigma(6) + 8, \sigma(6) + 12, \sigma(6) + 16\} = \emptyset$, implies $a \neq -2, -4, -6$. We consider the cases $a = -1$, $a = -3$, and $a \leq -5$ separately.

- If $a = -1$, then $\sigma(6) + 5$ and $\sigma(6) + 6$ are missing from $\sigma(1) \dots \sigma(10)$, which is a contradiction.
- If $a = -3$ then the two intervals $[\sigma(6) + 1, \sigma(5) - 1] = [\sigma(6) + 1, \sigma(6) + 2]$ and $[\sigma(3) + 1, \sigma(9) - 1] = [\sigma(6) + 10, \sigma(6) + 11]$ must each contain one of $\sigma(1)$ or $\sigma(2)$. In the second interval, whichever value is not achieved by $\sigma(1)$ or $\sigma(2)$ is followed by $\sigma(6) + 7$ and $\sigma(6) + 5$ in the suffix, which contradicts concavity.
- Next, consider $a \leq -5$, note also that $a \neq -6$. Then the two intervals $[\sigma(6) + 1, \sigma(6) + 3]$ and $[\sigma(6) + 5, \sigma(6) + 7]$ each must contain a value occurring at $\sigma(1)$ or $\sigma(2)$ since only $\sigma(5) \geq \sigma(6) + 5$ can have nonempty intersection with the second interval at $\sigma(6) + 5$ or $\sigma(6) + 7$. Hence, $\sigma(6) + 1 \leq \sigma(1), \sigma(2) \leq \sigma(6) + 7$ and $\sigma'(1) \geq \sigma'(2)$ imply that

$$15 \leq -3a \leq 2\sigma(2) - \sigma(1) - \sigma(6) \leq 13,$$

which is a contradiction.

If $b \geq 5$, then the intervals $[\sigma(6) + 1, \sigma(7) - 1]$, $[\sigma(7) + 1, \sigma(8) - 1]$, $[\sigma(8) + 1, \sigma(9) - 1]$, contain at least 4 consecutive integers. As before, concavity of the suffix implies that two consecutive integers below $\sigma(9)$ cannot both be missing from the first 10 entries $\sigma(1) \dots \sigma(10)$. But since these intervals have empty intersection with $\sigma(6) \dots \sigma(10)$ then a pair of entries from each of the three intervals must be achieved by the values $\sigma(1) \dots \sigma(5)$, which is a contradiction.

4. PERMUTATIONS WITH NO INFLECTIONS

The characterization of permutations with no inflections in Theorem 4.1 is equivalent to the classification of convex permutations by Brualdi and Dahl [BD20, Theorem 6]: a permutation has no inflection points if and only if it or its complement is convex. We give a different, inductive proof, and record the resulting count, which is novel.

Theorem 4.1. *Fix $n \geq 4$. A permutation $w \in \text{Sym}_n$ has no inflection points if and only if either w or a fundamental symmetry of w is one of the following permutations:*

- (A) $123 \dots n$,
- (B) $1345 \dots n2$,
- (C) $1357 \dots 8642$, or
- (D) $1n(n-1)(n-2) \dots 32$.

Then the number of permutations in Sym_n with zero inflection points is

$$\begin{cases} 10 & \text{if } n = 4, \text{ and} \\ 14 & \text{if } n \geq 5. \end{cases}$$

Proof. The fact that the listed permutations and their fundamental symmetries all have zero inflection points is easy to check. Fix $w \in \text{Sym}_n$ and assume that w has zero inflection points. We will prove that w or one of its fundamental symmetries has one of the listed forms. The result is easy to check in small cases, so fix $n \geq 5$ and we assume inductively that the result holds in Sym_m whenever $m \in [4, n-1]$.

Suppose, first, that $w''(i) = 0$ for all $i \in [2, n-1]$. Then, in particular, we must have $\{1, n\} = \{w(1), w(n)\}$, because otherwise $w''(w^{-1}(1)) > 0$ and/or $w''(w^{-1}(n)) < 0$. Suppose, without loss of generality, that $w(1) = 1$. Then, if $w(j) = 2$ for some $j > 2$, we will find that $w''(j) > 0$, which would be a contradiction. Proceeding in this way, we find that w must be the identity permutation. Had we instead assumed that $w(1) = n$, then w would have been the reverse of the identity permutation.

Now suppose that $w''(i) \neq 0$ for some $i \in [2, n-1]$. If $w(a) = 1$ and $w(b) = n$ for $a, b \notin \{1, n\}$, then we will have $w''(a) > 0 > w''(b)$. Thus we can assume, without loss of generality, that $w(1) = 1$. (The other three cases are fundamental symmetries of this choice.) Consider the permutation $v \in \text{Sym}_{n-1}$ defined by $v(j) := w(j+1) - 1$. Then $v''(j) = w''(j+1)$ for all $j \in [2, n-2]$. Therefore, by the inductive hypothesis, v (or a fundamental symmetry of it) must have one of the listed forms.

The remainder of the proof follows from a straightforward case analysis. We consider each possibility of v being a fundamental symmetry of one of the four listed forms, and analyze when w (obtained by appending a smallest element to the left end of that v) is acceptable; i.e., when w has zero inflection points:

- When v is of form (A), then w is also of that form. When v is the reverse of form (A), then w has form (D).
- When v is any fundamental symmetry of form (B), the resulting w will not be acceptable.

- When v is the reverse of form (C), then w has form (C). No other fundamental symmetries of this form will produce an acceptable w .
- When v is the reverse of form (D), then w has form (B). No other fundamental symmetries of this form will produce an acceptable w .

Form (A) permits two permutations (the complement of the identity always coincides with the reverse of the identity), form (B) permits four permutations, form (C) (which coincides with form (B) when $n = 4$) produces four permutations, and form (D) produces four permutations. This gives a total of 10 inflection point-free permutations in Sym_4 . When $n \geq 5$, none of the listed forms coincide, and there will always be $2 + 4 + 4 + 4 = 14$ inflection point-free permutations in Sym_n . $\square$

5. PERMUTATIONS WITH ONE INFLECTION

In this section, we address permutations of $[n]$ with one inflection. We determine, for every $2 \leq k < m$, all the values of n for which there exists $\sigma \in \text{Sym}_n$ with $\mathcal{I}(\sigma) = \{(k, m)\}$: see Theorem 5.10 for $m \geq 2k + 1$, and Theorem 5.14 for $k < m \leq 2k$. In particular, $I = \{(k, m)\}$ is always an admissible inflection set, and I is vanishing if and only if $m \geq 2k + 1$, see Corollary 5.15.

Lemma 5.1. *Let $\sigma \in \text{Sym}_n$ with $(k, m) \in \mathcal{I}(\sigma)$. Then $\sigma(k), \sigma(k+1), \dots, \sigma(m)$ must form an arithmetic progression.*

Proof. Since $\sigma''(t) = 0$ for all $k < t < m$, $\sigma(t-1) + \sigma(t+1) - 2\sigma(t) = 0$, hence

$$\sigma(t+1) - \sigma(t) = \sigma(t) - \sigma(t-1).$$

Let $d = \sigma(k+1) - \sigma(k) = \sigma(t+1) - \sigma(t)$ for $k < t < m$, so for $k \leq j \leq m$,

$$\sigma(j) = \sigma(k) + (j - k)d.$$

$\square$

Lemma 5.2. *Let $\sigma \in \text{Sym}_n$ with $\mathcal{I}(\sigma) = \{(k, m)\}$. By Lemma 5.1, we can say $\sigma(j) = \sigma(k) + (j - k)d$ for $k \leq j \leq m$. The complement satisfies $(\bar{\sigma})'' = -\sigma''$, hence $\mathcal{I}(\bar{\sigma}) = \mathcal{I}(\sigma)$, while the slope of the arithmetic progression changes sign. We may therefore assume $d > 0$. Under this assumption, we have $\sigma''(k) > 0$ and $\sigma''(m) < 0$.*

Proof. Given $\sigma \in \text{Sym}_n$, it is clear that $(\bar{\sigma})'' = -\sigma''$ and $\mathcal{I}(\bar{\sigma}) = \mathcal{I}(\sigma)$. Assume $\mathcal{I}(\sigma) = \{(k, m)\}$ and $d > 0$. Assume by contradiction that there exists ℓ_0 with $2 \leq \ell_0 < k$ and $\sigma''(\ell_0)\sigma''(k) < 0$. Let

$$i = \max\{\ell \in [2, k] : \sigma''(\ell)\sigma''(k) < 0\}$$

and

$$j = \min\{\ell \in (i, k] : \sigma''(\ell) \neq 0\} \leq k.$$

Then $\sigma''(i)\sigma''(j) < 0$ and $\sigma'' \equiv 0$ on (i, j) , so $(i, j) \in \mathcal{I}(\sigma)$ with $j \leq k < m$, a contradiction. Hence $\sigma''(\ell)\sigma''(k) \geq 0$ on $[2, k]$, and similarly $\sigma''(\ell)\sigma''(m) \geq 0$ on $[m, n-1]$.

Assume by contradiction that $\sigma''(k) < 0$ (hence $\sigma''(m) > 0$). Then σ' is non-increasing on $[1, k-1]$ and $\sigma'(k-1) > \sigma'(k) = d$, so $\sigma' > 0$ on $[1, k-1]$. Similarly, σ' is non-decreasing on $[m, n-1]$ with $\sigma'(m) > \sigma'(m-1) = d$, so $\sigma' > 0$ on $[m, n-1]$. As $\sigma' \equiv d > 0$ on $[k, m-1]$, we conclude that σ is strictly increasing, which forces $\sigma = e$. This implies $\sigma'' \equiv 0$ and $\mathcal{I}(\sigma) = \emptyset$, a contradiction. $\square$

The next lemma fully characterizes when $\mathcal{I}(\sigma) = \{(k, m)\}$. It will be used both to describe the permutations with a single inflection and to create permutations with a prescribed single inflection.

Lemma 5.3. Let $\sigma \in \text{Sym}_n$ and $2 \leq k < m \leq n - 1$. Let $d = \sigma(k + 1) - \sigma(k)$ and assume $d > 0$. Then $\mathcal{I}(\sigma) = \{(k, m)\}$ if and only if the following three conditions hold:

$$\sigma'(1) \leq \sigma'(2) \leq \dots \leq \sigma'(k - 1) < d = \sigma'(k), \quad (3)$$

$$\sigma'(j) = d \quad \text{for } k \leq j \leq m - 1, \quad (4)$$

$$d = \sigma'(m - 1) > \sigma'(m) \geq \sigma'(m + 1) \geq \dots \geq \sigma'(n - 1). \quad (5)$$

Proof. Assume first $\mathcal{I}(\sigma) = \{(k, m)\}$. By Lemma 5.1, we have $\sigma(j) = \sigma(k) + (j - k)d$ for $k \leq j \leq m$, that is, (4) holds. Since $d > 0$, Lemma 5.2 gives $\sigma''(k) > 0$ and $\sigma''(m) < 0$. Moreover, the proof of Lemma 5.2 shows that $\sigma''(\ell) \geq 0$ for $2 \leq \ell \leq k$ and $\sigma''(\ell) \leq 0$ for $m \leq \ell \leq n - 1$. Since $\sigma''(\ell) = \sigma'(\ell) - \sigma'(\ell - 1)$ and $\sigma'(k) = \sigma'(m - 1) = d$, this means that

$$\sigma'(1) \leq \sigma'(2) \leq \dots \leq \sigma'(k - 1) < \sigma'(k) = d,$$

$$d = \sigma'(m - 1) > \sigma'(m) \geq \sigma'(m + 1) \geq \dots \geq \sigma'(n - 1),$$

that is, (3) and (5) hold.

Reciprocally, assume (3)–(5) and let us prove that $\mathcal{I}(\sigma) = \{(k, m)\}$. Since $\sigma''(\ell) = \sigma'(\ell) - \sigma'(\ell - 1)$, (3)–(5) yield

- $\sigma''(\ell) \geq 0$ for $2 \leq \ell < k$,
- $\sigma''(k) > 0$,
- $\sigma''(\ell) = 0$ for $k < \ell < m$,
- $\sigma''(m) < 0$,
- $\sigma''(\ell) \leq 0$ for $m < \ell \leq n - 1$.

In particular, $(k, m) \in \mathcal{I}(\sigma)$. Let now $(i, j) \in \mathcal{I}(\sigma)$, so that $i < j$ and $\sigma''(i)\sigma''(j) < 0$. Since σ'' is positive only at indices in $[2, k]$ and negative only at indices in $[m, n - 1]$, and since $i < j$, we must have $i \leq k$ and $j \geq m$. If $i < k$, then $i < k < j$, but $\sigma''(k) \neq 0$, contradicting $\sigma''(\ell) = 0$ for $i < \ell < j$. Therefore, $i = k$. If $j > m$, then $i < m < j$ but $\sigma''(m) \neq 0$, leading to the same contradiction. Therefore, $j = m$ as well. $\square$

Let $2 \leq k < m$, $n \geq m + 1$ and $\sigma \in \text{Sym}_n$ with $\mathcal{I}(\sigma) = \{(k, m)\}$. By Lemma 5.1, $\sigma(j) = \sigma(k) + (j - k)d$ for $k \leq j \leq m$, where $d = \sigma(k + 1) - \sigma(k)$. By Lemma 5.2, we may assume $d > 0$, in which case $\sigma''(k) > 0$ and $\sigma''(m) < 0$. Using Lemma 5.3 and the fact that σ' takes integer values, we conclude that $\sigma'(j) \leq d - 1$ for $1 \leq j \leq k - 1$ and for $m \leq j \leq n - 1$.

The next lemma describes the shape of a permutation with a single inflection.

Lemma 5.4. Let $2 \leq k < m$ and $n \geq m + 1$. Let $\sigma \in \text{Sym}_n$ with $\mathcal{I}(\sigma) = \{(k, m)\}$ and $d = \sigma(k + 1) - \sigma(k) > 0$. Then there exist $k^* \in [1, k]$ and $m^* \in [m, n]$ so that

- (a) • σ is strictly decreasing on $[1, k^*]$, and
- σ is strictly increasing on $[k^*, m^*]$, and
- σ is strictly decreasing on $[m^*, n]$,
- with $0 < \sigma'(j) \leq d - 1$ for $j \in [k^*, k - 1] \cup [m, m^* - 1]$.
- (b) • $|\sigma'|$ is non-increasing on $[1, k^* - 1]$ and
- $|\sigma'|$ is non-decreasing on $[m^*, n - 1]$.
- (c) $\{1, \dots, \sigma(k^*) - 1\} \subseteq \sigma([m^* + 1, n])$ and $\{\sigma(m^*) + 1, \dots, n\} \subseteq \sigma([1, k^* - 1])$.
- (d) Moreover, $\sigma(k^*) \leq 2$ and $\sigma(m^*) \geq n - 1$.

Proof. By (3), either there exists $1 \leq k^* \leq k - 1$ so that $\{j \in [1, k - 1] : \sigma'(j) > 0\} = [k^*, k - 1]$, or $\{j \in [1, k - 1] : \sigma'(j) > 0\} = \emptyset$ (in this case, we will write $k^* = k$). Similarly, by (5),

either there exists $m^* \in [m+1, n]$ so that $\{j \in [m, n-1] : \sigma'(j) > 0\} = [m, m^* - 1]$ or $\{j \in [m, n-1] : \sigma'(j) > 0\} = \emptyset$ (in this case, we will write $m^* = m$). This implies that

- σ is strictly decreasing on $[1, k^*]$,
- σ is strictly increasing on $[k^*, m^*]$, and
- σ is strictly decreasing on $[m^*, n]$,

with $0 < \sigma'(j) \leq d-1$ for $j \in [k^*, k^* - 1] \cup [m, m^* - 1]$. This proves (a).

Given $j \in [m^*, n-1]$, $\sigma'(j) < 0$ and, by (5), σ' is non-increasing in $[m^*, n-1]$. Similarly, given $j \in [1, k^* - 1]$, $\sigma'(j) < 0$, and σ' is non-decreasing by (3). We conclude that

- $|\sigma'|$ is non-decreasing on $[m^*, n-1]$,
- $|\sigma'|$ is non-increasing on $[1, k^* - 1]$,

which is (b).

Given $j \in [k^*, m^*]$, $\sigma(j) \geq \sigma(k^*)$, hence $\{1, \dots, \sigma(k^*) - 1\} \cap \{\sigma(k^*), \dots, \sigma(m^*)\} = \emptyset$. Moreover, $\{1, \dots, \sigma(k^*) - 1\} \cap \{\sigma(1), \dots, \sigma(k^* - 1)\} = \emptyset$, as σ is strictly decreasing on $[1, k^*]$. This implies that $\{1, \dots, \sigma(k^*) - 1\} \subseteq \{\sigma(m^* + 1), \dots, \sigma(n)\}$, and $\{1, \dots, \sigma(k^*) - 1\}$ are the smallest values in this last third of the permutation.

Assume by contradiction that $\sigma(k^*) \geq 3$. Since $\{1, \dots, \sigma(k^*) - 1\} \subseteq \{\sigma(m^* + 1), \dots, \sigma(n)\}$ and σ is strictly decreasing on $[m^*, n]$, $|\sigma'(n-1)| = 1$. Since $|\sigma'|$ is non-decreasing on $[m^*, n-1]$, this shows that $\{\sigma(m^*), \dots, \sigma(n)\} = \{1, \dots, \sigma(m^*)\}$. Now, since $1 \leq \sigma(k^*) < \sigma(m^*)$, $\sigma(k^*) \in \{\sigma(m^*), \dots, \sigma(n)\}$. However, $k^* \leq k < m \leq m^*$, contradicting the fact that σ is a permutation (hence injective). We conclude that $\sigma(k^*) \leq 2$.

Now, given $j \in [m^* + 1, n]$, $\sigma(j) < \sigma(m^*)$, hence $\{\sigma(m^*) + 1, \dots, n\} \cap \{\sigma(m^* + 1), \dots, \sigma(n)\} = \emptyset$. Moreover, $\{\sigma(m^*) + 1, \dots, n\} \cap \{\sigma(k^*), \dots, \sigma(m^*)\} = \emptyset$, as σ is strictly increasing on $[k^*, m^*]$. Hence $\{\sigma(m^*) + 1, \dots, n\} \subseteq \{\sigma(1), \dots, \sigma(k^* - 1)\}$, and $\{\sigma(m^*) + 1, \dots, n\}$ are the largest values of this first third of the permutation.

Assume by contradiction that $\sigma(m^*) \leq n-2$. Then $\{n-1, n\} \subseteq \{\sigma(1), \dots, \sigma(k^* - 1)\}$. Since σ is strictly decreasing on $[1, k^*]$, we conclude that $\sigma(1) = n$ and $\sigma(2) = n-1$. Since $|\sigma'|$ is non-increasing on $[1, k^* - 1]$, we conclude that $\{\sigma(1), \dots, \sigma(k^* - 1), \sigma(k^*)\} = \{\sigma(k^*), \dots, n\}$. We also have $\sigma(m^*) \in \{\sigma(k^*), \dots, n\}$, but $m^* \notin [1, k^*]$, contradicting the injectivity of σ . Therefore $\sigma(m^*) \geq n-1$. This proves (c) and (d). $\square$

From now on, given σ as in Lemma 5.4, we fix k^* and m^* as in that lemma, and we call

$$r = k - k^* \geq 0, \quad p = m^* - m \geq 0.$$

We will refer to $\sigma([1, k^* - 1])$ as the first third of σ and to $\sigma([m^* + 1, n])$ as the last third of σ . Notice that they have $k-1-r$ and $n-m-p$ elements, respectively. The set of values that σ misses within the inflection is

$$H_{infl} = \{\sigma(k) + id + j : 0 \leq i \leq m - k - 1, 1 \leq j \leq d-1\}.$$

Since σ is strictly increasing on $[k^*, m^*]$ and, for each $0 \leq i \leq m - k - 1$, $\sigma(k) + id = \sigma(k+i)$ and $\sigma(k) + (i+1)d = \sigma(k+i+1)$, we have $H_{infl} \cap \sigma([k^*, m^*]) = \emptyset$. Therefore

$$H_{infl} \subseteq \sigma([1, k^* - 1]) \cup \sigma([m^* + 1, n]). \quad (6)$$

We now analyze the possible values of d separately.

Lemma 5.5. *Let σ be as in Lemma 5.4. If $d = 1$, then $k = 2$, $n = m+1$ and $\sigma = (m+1)23 \dots m1$, see Figure 3.*

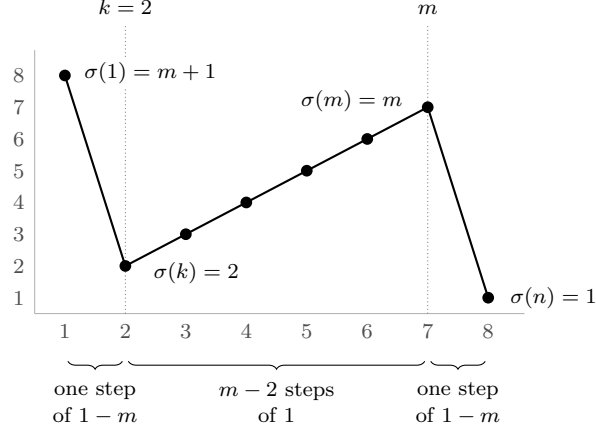


FIGURE 3. The permutation $\sigma = (m+1)23\dots m1$ of Lemma 5.5 when $m = 7$, that is, $\sigma = 82345671$. Since $d = 1$, the progression $\sigma(k), \dots, \sigma(m)$ skips no values, so the only entries outside of it are n , in the first position, and 1, in the last one.

Proof. Since $d = 1$, Lemma 5.4(a) shows that there is no j in $[k^*, k-1] \cup [m, m^*-1]$, hence $k^* = k$ and $m^* = m$, and $\sigma([k, m]) = \{\sigma(k), \sigma(k) + 1, \dots, \sigma(m)\}$.

Since $k \geq 2$ and $\sigma'(k-1) < 0$, we have $\sigma(k-1) > \sigma(k)$. As $\sigma(k-1) \notin \sigma([k, m])$, this gives $\sigma(k-1) \geq \sigma(m) + 1$. By Lemma 5.4(d), $\sigma(m) \geq n-1$, so $\sigma(k-1) = n$. If $k \geq 3$, then $\sigma(k-2) > \sigma(k-1) = n$, as σ is strictly decreasing on $[1, k]$, which is a contradiction. Hence $k = 2$.

Similarly, since $n \geq m+1$ and $\sigma'(m) < 0$, $\sigma(m+1) < \sigma(m)$, and $\sigma(m+1) \notin \sigma([k, m])$ gives $\sigma(m+1) \leq \sigma(k) - 1$. By Lemma 5.4(d), $\sigma(k) \leq 2$, so $\sigma(m+1) = 1$ and $\sigma(k) = 2$. If $n \geq m+2$, then $\sigma(m+2) < \sigma(m+1) = 1$, which is a contradiction. Hence $n = m+1$.

We conclude that $\sigma(1) = n = m+1$, $\sigma(j) = j$ for $2 \leq j \leq m$, and $\sigma(m+1) = 1$. $\square$

Lemma 5.6. *Let σ be as in Lemma 5.4. If $d \geq 3$, then $m \leq 2k$ and $n \geq 2m - k - 1$.*

Proof. For each $0 \leq i \leq m - k - 1$, let

$$H_i = \{\sigma(k) + id + 1, \dots, \sigma(k) + id + (d-1)\},$$

so that H_{infl} is the disjoint union of $H_0, \dots, H_{m-k-1}$, see Figure 4.

Since $d \geq 3$, each H_i consists of $d-1 \geq 2$ consecutive integers.

We first claim that if $H_i \subseteq \sigma([1, k^*-1])$, then $i = 0$. Indeed, let $b, b+1 \in H_i$. Since σ is strictly decreasing on $[1, k^*]$, there exists $j \in [1, k^*-2]$ so that $\sigma(j) = b+1$ and $\sigma(j+1) = b$, that is, $|\sigma'(j)| = 1$. Since $|\sigma'|$ is non-increasing on $[1, k^*-1]$, we conclude that $\{\sigma(j), \dots, \sigma(k^*)\} = \{\sigma(k^*), \dots, b+1\}$. Now, $\sigma(k^*) \leq \sigma(k) \leq \sigma(k) + id < b$, hence $\sigma(k+i) = \sigma(k) + id \in \{\sigma(j), \dots, \sigma(k^*)\}$. By injectivity of σ , $k+i \leq k^* \leq k$, that is, $i = 0$.

Next we claim that if $H_i \subseteq \sigma([m^*+1, n])$, then $i = m - k - 1$. Indeed, let $b, b+1 \in H_i$. Since σ is strictly decreasing on $[m^*, n]$, there exists $j \in [m^*+1, n-1]$ so that $\sigma(j) = b+1$ and $\sigma(j+1) = b$, that is, $|\sigma'(j)| = 1$. Since $|\sigma'|$ is non-decreasing on $[m^*, n-1]$, we conclude that $\{\sigma(m^*), \dots, \sigma(j+1)\} = \{b, \dots, \sigma(m^*)\}$. Now, $b < \sigma(k) + (i+1)d \leq \sigma(m) \leq \sigma(m^*)$, hence $\sigma(k+i+1) = \sigma(k) + (i+1)d \in \{\sigma(m^*), \dots, \sigma(j+1)\}$. By injectivity of σ , $k+i+1 \geq m^* \geq m$, that is, $i = m - k - 1$.

By (6) and the first claim, each of the $m - k - 1$ disjoint sets $H_1, \dots, H_{m-k-1}$ contains an element of $\sigma([m^*+1, n])$. Hence $n - m^* \geq m - k - 1$, and so $n \geq m^* + m - k - 1 \geq 2m - k - 1$. By

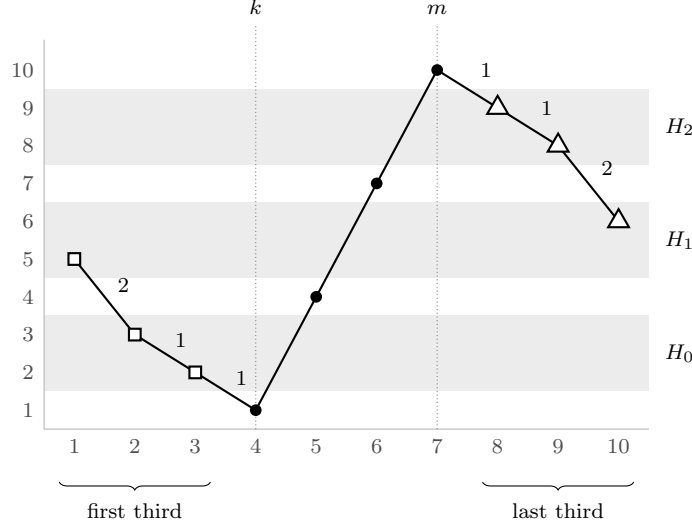


FIGURE 4. Lemma 5.6 with $\sigma = 5\ 3\ 2\ 1\ 4\ 7\ 10\ 9\ 8\ 6$, where $(k, m, n) = (4, 7, 10)$ and $d = 3$. The shaded bands are the sets $H_0 = \{2, 3\}$, $H_1 = \{5, 6\}$ and $H_2 = \{8, 9\}$, of values skipped by the progression. The numbers next to the graph are the values of $|\sigma'|$ in the first and last thirds.

(6) and the second claim, each of the $m - k - 1$ disjoint sets $H_0, \dots, H_{m-k-2}$ contains an element of $\sigma([1, k^* - 1])$. Hence $m - k - 1 \leq k^* - 1 \leq k - 1$, that is, $m \leq 2k$. $\square$

We are left with the case $d = 2$. In this case, the progression $\sigma(k), \dots, \sigma(m)$ skips exactly one value between any two consecutive entries, so it skips $\ell = m - k$ values $h_1 < \dots < h_\ell$. By (6), each of them appears in the first third or in the last third of σ . We mark this choice in two sets of indices, A and B . We will see that σ is completely described by A and B together with four numbers: the lengths r and p of the two parts next to the progression where σ increases by 1, and two numbers $a, c \in \{0, 1\}$, which tell us whether n is the first entry of σ and whether 1 is the last one (see Figure 6 for an example). Counting entries gives two relations between these parameters, k and n , namely (9) and (10), and (13) allows us to compute $|\sigma'(j)|$ for $j \in [1, k^* - 1] \cup [m^*, n - 1]$, in terms of the parameters. By Lemma 5.4(b), these values are non-increasing on $[1, k^* - 1]$ and non-decreasing on $[m^*, n - 1]$, and this is what restricts the parameters in the proofs below.

So, let σ be as in Lemma 5.4, assume $d = 2$, and write $\ell = m - k$. By Lemma 5.4(a), $\sigma'(j) = 1$ for $j \in [k^*, k - 1] \cup [m, m^* - 1]$, hence

$$\sigma(k^*) = \sigma(k) - r, \quad \sigma(m^*) = \sigma(m) + p = \sigma(k) + 2\ell + p, \quad (7)$$

and

$$\sigma([k^*, m^*]) = \{\sigma(k) - r, \dots, \sigma(k)\} \cup \{\sigma(k) + 2i : 0 \leq i \leq \ell\} \cup \{\sigma(m), \dots, \sigma(m) + p\}.$$

In this case $H_{infl} = \{h_1, \dots, h_\ell\}$, where

$$h_i = \sigma(k) + 2i - 1, \quad 1 \leq i \leq \ell,$$

and the set of values that σ misses on $[k^*, m^*]$ is $\{1, \dots, \sigma(k^*) - 1\} \cup H_{infl} \cup \{\sigma(m^*) + 1, \dots, n\}$. Let

$$A = \{i \in [1, \ell] : h_i \in \sigma([1, k^* - 1])\}, \quad B = \{i \in [1, \ell] : h_i \in \sigma([m^* + 1, n])\}.$$

By (6), $[1, \ell]$ is the disjoint union of A and B . Recall that the values in $\sigma([1, k^* - 1])$ are larger than $\sigma(k^*)$, and the values in $\sigma([m^* + 1, n])$ are smaller than $\sigma(m^*)$. Together with Lemma 5.4(c),

this gives

$$\begin{aligned}\sigma([1, k^* - 1]) &= \{h_i : i \in A\} \cup \{\sigma(m^*) + 1, \dots, n\}, \\ \sigma([m^* + 1, n]) &= \{h_i : i \in B\} \cup \{1, \dots, \sigma(k^*) - 1\}.\end{aligned}\tag{8}$$

By Lemma 5.4(d), the numbers

$$a = n - \sigma(m^*) \quad \text{and} \quad c = \sigma(k^*) - 1$$

belong to $\{0, 1\}$. Let $q = |A|$ and $t = |B| = \ell - q$, and write $A = \{i_1 > i_2 > \dots > i_q\}$ and $B = \{j_1 > j_2 > \dots > j_t\}$. Counting the elements of both sets in (8), we obtain $k^* - 1 = q + a$ and $n - m^* = t + c$, that is,

$$q + a + r = k - 1,\tag{9}$$

$$n = m + p + \ell - q + c = 2\ell + k + p + c - q.\tag{10}$$

Since σ is strictly decreasing on $[1, k^*]$ and on $[m^*, n]$, (8) determines the order of the entries in the first and last thirds of σ . Recall that $k^* - 1 = q + a$ and $n - m^* = t + c$.

In the first third, the values $h_{i_1} > h_{i_2} > \dots > h_{i_q}$ appear in this order, and they are preceded by n when $a = 1$. That is,

$$\sigma(a + s) = h_{i_s} \quad \text{for } 1 \leq s \leq q, \quad \text{and} \quad \sigma(1) = n \quad \text{if } a = 1.\tag{11}$$

In the last third, the values $h_{j_1} > h_{j_2} > \dots > h_{j_t}$ appear in this order, and they are followed by 1 when $c = 1$. That is,

$$\sigma(m^* + s) = h_{j_s} \quad \text{for } 1 \leq s \leq t, \quad \text{and} \quad \sigma(n) = 1 \quad \text{if } c = 1.\tag{12}$$

In particular, $\sigma(k^* - 1) = h_{i_q}$ if $q \geq 1$, and $\sigma(m^* + 1) = h_{j_1}$ if $t \geq 1$. Finally, by (7) and the definitions of a and c , for $i, i' \in [1, \ell]$ we have

$$\begin{aligned}h_i - h_{i'} &= 2(i - i'), & h_i - \sigma(k^*) &= 2i - 1 + r, & \sigma(m^*) - h_i &= 2(\ell - i) + p + 1, \\ n - h_i &= 2(\ell - i) + p + 1 + a, & h_i - 1 &= 2i + r + c - 1.\end{aligned}\tag{13}$$

The next lemma shows that the sets A and B are very rigid, see Figure 5. If two neighbouring skipped values h_{i-1} and h_i are both used in the first third, they are consecutive entries of σ and differ by 2. Since $|\sigma'|$ is non-increasing there, from then on σ can only decrease by 2 at a time, which forces all the smaller skipped values to be used in the first third as well; this is (a), and (b) is the symmetric statement for the last third. As a consequence, the elements of A , and those of B , cannot be far apart, which is (c). Finally, (d) and (e) compare the difference between the two entries next to the increasing part of σ , which grows with r (respectively, with p), with the neighbouring difference, which is at most 4 by (c). This bounds r and p .

Lemma 5.7. *Let σ be as in Lemma 5.4 with $d = 2$, and let $\ell, h_i, A, B, q, t, a, c$ be as above.*

- (a) *If $i \geq 2$ and $i, i - 1 \in A$, then $\{1, \dots, i\} \subseteq A$.*
- (b) *If $j \geq 2$ and $j, j - 1 \in B$, then $\{j - 1, \dots, \ell\} \subseteq B$.*
- (c) *Any two consecutive elements of A differ by at most 2, and any two consecutive elements of B differ by at most 2.*
- (d) *If $q \geq 2$, then $2i_q - 1 + r \leq 2(i_{q-1} - i_q) \leq 4$. In particular, $r \leq 3$, and if $r \geq 2$, then $i_q = 1$ and $i_{q-1} = 3$.*
- (e) *If $t \geq 2$, then $2(\ell - j_1) + p + 1 \leq 2(j_1 - j_2) \leq 4$. In particular, $p \leq 3$. If $j_1 - j_2 = 1$, then $p \leq 1$. If $p \geq 2$, then $B = \{\ell, \ell - 2, \ell - 4, \dots, \ell - 2(t - 1)\}$.*

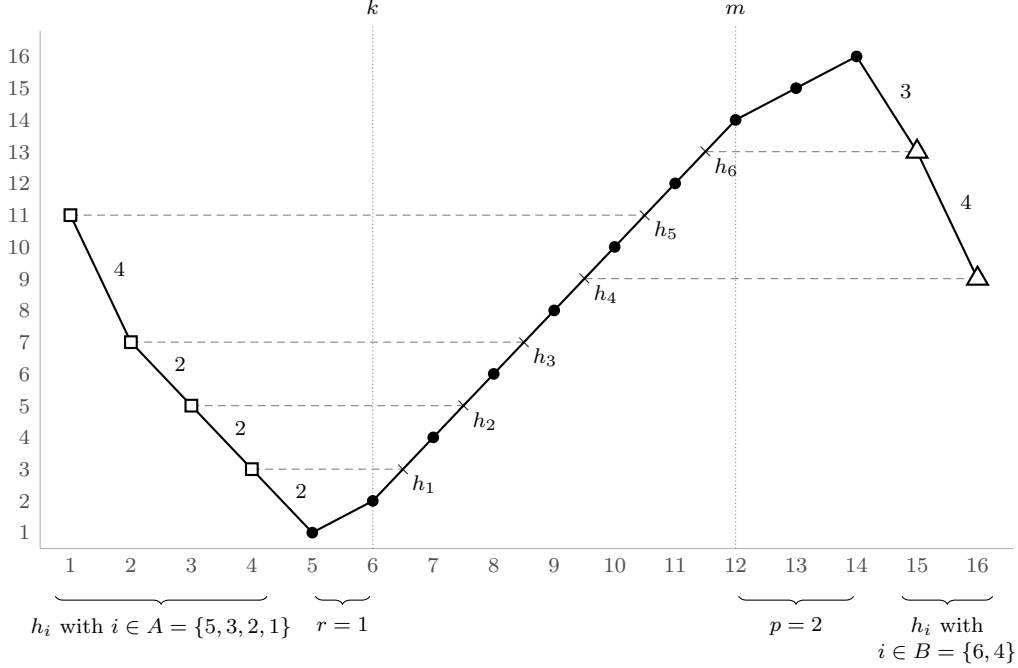


FIGURE 5. Lemma 5.7 with $\sigma = 11\ 7\ 5\ 3\ 1\ 2\ 4\ 6\ 8\ 10\ 12\ 14\ 15\ 16\ 13\ 9$, where $(k, m, n) = (6, 12, 16)$, $\ell = 6$ and $h_i = 2i + 1$. The numbers next to the graph are the values of $|\sigma'|$ in the first and last thirds. Since $3, 2 \in A$, the values h_3 and h_2 are consecutive entries and $|\sigma'(2)| = 2$. From then on, $|\sigma'| \leq 2$ in the first third, which forces h_1 to be the next entry, as in (a). Any two consecutive elements of A , and of B , differ by at most 2, as in (c). The last value of $|\sigma'|$ in the first third is $2i_q - 1 + r = 2$, and the first one in the last third is $2(\ell - j_1) + p + 1 = 3$, as in (d) and (e). Furthermore, $p = 2$ and $B = \{\ell, \ell - 2\}$.

Proof. Throughout the proof we will use the following three facts: first, by (6), no h_i belongs to $\sigma([k^*, m^*])$. In particular, $h_i \neq \sigma(k^*)$ and $h_i \neq \sigma(m^*)$. Second, for $2 \leq i \leq \ell$, the only integer strictly between h_{i-1} and h_i is

$$h_i - 1 = h_{i-1} + 1 = \sigma(k) + 2(i-1) = \sigma(k+i-1), \quad \text{where } k+1 \leq k+i-1 \leq m-1. \quad (14)$$

Third, by Lemma 5.4(b), $|\sigma'|$ is non-increasing on $[1, k^* - 1]$ and non-decreasing on $[m^*, n - 1]$.

Proof of (a): Let $i \geq 2$ with $i, i-1 \in A$, that is, $h_i, h_{i-1} \in \sigma([1, k^* - 1])$.

We first show that h_i and h_{i-1} appear in consecutive positions. Indeed, σ is strictly decreasing on $[1, k^*]$, and by (14) the only integer strictly between h_{i-1} and h_i is $\sigma(k+i-1)$, which is not in $\sigma([1, k^* - 1])$ because $k+i-1 > k^*$. Hence there exists $s \in [1, k^* - 2]$ so that

$$\sigma(s) = h_i \quad \text{and} \quad \sigma(s+1) = h_{i-1}.$$

In particular, $|\sigma'(s)| = h_i - h_{i-1} = 2$. Since $|\sigma'|$ is non-increasing on $[1, k^* - 1]$, we conclude that

$$|\sigma'(u)| \leq 2 \quad \text{for all } s \leq u \leq k^* - 1. \quad (15)$$

Next we claim the following: if $u \in [s, k^* - 1]$ and $\sigma(u) = h_{i'}$ for some $i' \geq 2$, then $\sigma(u+1) = h_{i'-1}$ (and therefore $u+1 \leq k^* - 1$, since $h_{i'-1} \neq \sigma(k^*)$). Indeed, $\sigma(u+1) < \sigma(u)$ and, by (15), $\sigma(u) - \sigma(u+1) \leq 2$, so

$$\sigma(u+1) \in \{h_{i'} - 1, h_{i'} - 2\}.$$

By (14), $h_{i'} - 1 = \sigma(k + i' - 1)$ with $k + i' - 1 \geq k + 1 > k^* \geq u + 1$, so the injectivity of σ gives $\sigma(u + 1) \neq h_{i'} - 1$. Therefore $\sigma(u + 1) = h_{i'} - 2 = h_{i'-1}$, which proves the claim.

Applying the claim repeatedly, starting from $\sigma(s + 1) = h_{i-1}$, we obtain

$$\sigma(s) = h_i, \quad \sigma(s + 1) = h_{i-1}, \quad \sigma(s + 2) = h_{i-2}, \quad \dots, \quad \sigma(s + i - 1) = h_1,$$

with $s + i - 1 \leq k^* - 1$. Hence $h_1, \dots, h_i \in \sigma([1, k^* - 1])$, that is, $\{1, \dots, i\} \subseteq A$.

Proof of (b): The argument is symmetric. Let $j \geq 2$ with $j, j - 1 \in B$, that is, $h_j, h_{j-1} \in \sigma([m^* + 1, n])$.

Since σ is strictly decreasing on $[m^*, n]$, and by (14) the only integer strictly between h_{j-1} and h_j is $\sigma(k + j - 1)$, which is not in $\sigma([m^* + 1, n])$ because $k + j - 1 < m^*$, there exists $s \in [m^* + 1, n - 1]$ so that

$$\sigma(s) = h_j \quad \text{and} \quad \sigma(s + 1) = h_{j-1}.$$

In particular, $|\sigma'(s)| = 2$. Since $|\sigma'|$ is non-decreasing on $[m^*, n - 1]$, we conclude that

$$|\sigma'(u)| \leq 2 \quad \text{for all } m^* \leq u \leq s. \quad (16)$$

Next we claim the following: if $u \in [m^* + 1, s]$ and $\sigma(u) = h_{j'}$ for some $j' \leq \ell - 1$, then $\sigma(u - 1) = h_{j'+1}$ (and therefore $u - 1 \geq m^* + 1$, since $h_{j'+1} \neq \sigma(m^*)$). Indeed, $\sigma(u - 1) > \sigma(u)$ and, by (16), $\sigma(u - 1) - \sigma(u) \leq 2$, so

$$\sigma(u - 1) \in \{h_{j'} + 1, h_{j'} + 2\}.$$

By (14), $h_{j'} + 1 = \sigma(k + j')$ with $k + j' \leq m - 1 < m^* \leq u - 1$, so the injectivity of σ gives $\sigma(u - 1) \neq h_{j'} + 1$. Therefore $\sigma(u - 1) = h_{j'} + 2 = h_{j'+1}$, which proves the claim.

Applying the claim repeatedly, starting from $\sigma(s) = h_j$, we obtain

$$\sigma(s + 1) = h_{j-1}, \quad \sigma(s) = h_j, \quad \sigma(s - 1) = h_{j+1}, \quad \dots, \quad \sigma(s - (\ell - j)) = h_\ell,$$

with $s - (\ell - j) \geq m^* + 1$. Hence $h_{j-1}, h_j, \dots, h_\ell \in \sigma([m^* + 1, n])$, that is, $\{j - 1, \dots, \ell\} \subseteq B$.

Proof of (c): Recall that $[1, \ell]$ is the disjoint union of A and B .

Assume by contradiction that $i > i'$ are consecutive elements of A with $i - i' \geq 3$. Then $i - 1$ and $i - 2$ are not in A , so $i - 1, i - 2 \in B$. By (b) (applied with $j = i - 1$), $\{i - 2, \dots, \ell\} \subseteq B$. In particular $i \in B$, contradicting $i \in A$.

Similarly, assume by contradiction that $j > j'$ are consecutive elements of B with $j - j' \geq 3$. Then $j - 1$ and $j - 2$ are not in B , so $j - 1, j - 2 \in A$. By (a) (applied with $i = j - 1$), $\{1, \dots, j - 1\} \subseteq A$. In particular $j' \in A$, contradicting $j' \in B$.

Proof of (d): Assume $q \geq 2$. Since $a + q = k^* - 1$, (11) (with $s = q - 1$ and $s = q$) gives

$$\sigma(k^* - 2) = h_{i_{q-1}} \quad \text{and} \quad \sigma(k^* - 1) = h_{i_q}.$$

By (13),

$$|\sigma'(k^* - 2)| = h_{i_{q-1}} - h_{i_q} = 2(i_{q-1} - i_q) \quad \text{and} \quad |\sigma'(k^* - 1)| = h_{i_q} - \sigma(k^*) = 2i_q - 1 + r.$$

Since $|\sigma'|$ is non-increasing on $[1, k^* - 1]$, $|\sigma'(k^* - 1)| \leq |\sigma'(k^* - 2)|$. Moreover, $i_{q-1} - i_q \leq 2$ by (c). Therefore

$$2i_q - 1 + r \leq 2(i_{q-1} - i_q) \leq 4. \quad (17)$$

Since $i_q \geq 1$, (17) gives $r \leq 5 - 2i_q \leq 3$.

Assume now $r \geq 2$. Then (17) gives $2i_q \leq 5 - r \leq 3$, hence $i_q = 1$. Using $i_q = 1$ in (17), we obtain $2(i_{q-1} - 1) \geq 1 + r \geq 3$, hence $i_{q-1} \geq 3$. Since $i_{q-1} - i_q \leq 2$, we conclude that $i_{q-1} = 3$.

Proof of (e): assume $t \geq 2$. By (12) (with $s = 1$ and $s = 2$),

$$\sigma(m^* + 1) = h_{j_1} \quad \text{and} \quad \sigma(m^* + 2) = h_{j_2}.$$

By (13),

$$|\sigma'(m^*)| = \sigma(m^*) - h_{j_1} = 2(\ell - j_1) + p + 1 \quad \text{and} \quad |\sigma'(m^* + 1)| = h_{j_1} - h_{j_2} = 2(j_1 - j_2).$$

Since $|\sigma'|$ is non-decreasing on $[m^*, n - 1]$, $|\sigma'(m^*)| \leq |\sigma'(m^* + 1)|$. Moreover, $j_1 - j_2 \leq 2$ by (c). Therefore

$$2(\ell - j_1) + p + 1 \leq 2(j_1 - j_2) \leq 4. \quad (18)$$

Since $j_1 \leq \ell$, (18) gives $p + 1 \leq 4$, that is, $p \leq 3$. If $j_1 - j_2 = 1$, (18) gives $p + 1 \leq 2$, that is, $p \leq 1$.

Assume now $p \geq 2$. Then the left-hand side of (18) is at least 3, so $2(j_1 - j_2) \geq 3$, hence $j_1 - j_2 = 2$. Using this in (18), we obtain $2(\ell - j_1) \leq 3 - p \leq 1$, hence $j_1 = \ell$. It remains to show that any two consecutive elements of B differ by exactly 2. Since $|\sigma'|$ is non-decreasing on $[m^*, n - 1]$,

$$|\sigma'(u)| \geq |\sigma'(m^* + 1)| = 2(j_1 - j_2) = 4 \quad \text{for all } m^* + 1 \leq u \leq n - 1.$$

For $1 \leq v \leq t - 1$, (12) gives $\sigma(m^* + v) = h_{j_v}$ and $\sigma(m^* + v + 1) = h_{j_{v+1}}$, so $2(j_v - j_{v+1}) = |\sigma'(m^* + v)| \geq 4$, that is, $j_v - j_{v+1} \geq 2$. By (c), $j_v - j_{v+1} = 2$. Since $j_1 = \ell$, we conclude that $B = \{\ell, \ell - 2, \ell - 4, \dots, \ell - 2(t - 1)\}$. $\square$

To construct permutations with slope $d = 2$ we will use Lemma 5.8 below. The idea is to first write the increasing part of σ (blocks (B3)-(B5)): a progression with $d = 2$, preceded by r steps of 1 and followed by p steps of 1. The ℓ values $h_1, \dots, h_\ell$ skipped by the progression are then split into two sets, which are written in decreasing order before and after the increasing part (blocks (B2) and (B6)), possibly together with n in the first position and 1 in the last one (blocks (B1) and (B7)). By Lemma 5.3, the only condition left to check is that the drops α_s at the beginning get smaller, and the drops β_s at the end get larger, see Figure 6.

Lemma 5.8. *Let $k \geq 2$, $\ell \geq 1$ and $m = k + \ell$. Let $a, c \in \{0, 1\}$ and $r, p \geq 0$ be integers, and let $[1, \ell]$ be the disjoint union of $A = \{i_1 > \dots > i_q\}$ and $B = \{j_1 > \dots > j_t\}$, where $t \geq 1$ and $q + a + r = k - 1$. Let*

$$x = c + r + 1, \quad h_i = x + 2i - 1 \quad \text{for } 1 \leq i \leq \ell, \quad n = x + 2\ell + p + a.$$

Let σ be obtained by listing, one after the other, the following seven blocks:

(B1) the single entry n if $a = 1$, and nothing if $a = 0$,

(B2) $h_{i_1} h_{i_2} \dots h_{i_q}$, which decreases,

(B3) $(c + 1)(c + 2) \dots x$, which has $r + 1$ entries and increases by 1,

(B4) $(x + 2)(x + 4) \dots (x + 2\ell)$, which has ℓ entries and increases by 2,

(B5) $(x + 2\ell + 1)(x + 2\ell + 2) \dots (x + 2\ell + p)$, which has p entries and increases by 1,

(B6) $h_{j_1} h_{j_2} \dots h_{j_t}$, which decreases,

(B7) the single entry 1 if $c = 1$, and nothing if $c = 0$.

Then $\sigma \in \text{Sym}_n$, with $\sigma(k) = x$ and $\sigma(m) = x + 2\ell$, and $n = 2\ell + r + a + p + c + 1$. Moreover, the numbers

$$\begin{aligned} \alpha_s &= \sigma(s) - \sigma(s + 1) & \text{for } 1 \leq s \leq q + a, \\ \beta_s &= \sigma(m + p + s - 1) - \sigma(m + p + s) & \text{for } 1 \leq s \leq t + c, \end{aligned}$$

are positive, and

$$\sigma' = \underbrace{(-\alpha_1) \dots (-\alpha_{q+a})}_{q+a} \underbrace{1 \dots 1}_r \underbrace{2 \dots 2}_\ell \underbrace{1 \dots 1}_p \underbrace{(-\beta_1) \dots (-\beta_{t+c})}_{t+c}. \quad (19)$$

If

$$\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{q+a} \quad \text{and} \quad \beta_1 \leq \beta_2 \leq \dots \leq \beta_{t+c}, \quad (20)$$

then $\mathcal{I}(\sigma) = \{(k, m)\}$.

Example 5.9. Let $\ell = 5$, $a = c = 1$, $r = 2$, $p = 1$, $A = \{3, 1\}$ and $B = \{5, 4, 2\}$, so that $k = 6$, $m = 11$, $x = 4$, $h_i = 2i + 3$ and $n = 16$, see Figure 6. Then

$$\sigma = \underbrace{16}_{(B1)} \underbrace{9 \ 5}_{(B2)} \underbrace{2 \ 3 \ 4}_{(B3)} \underbrace{6 \ 8 \ 10 \ 12 \ 14}_{(B4)} \underbrace{15}_{(B5)} \underbrace{13 \ 11 \ 7}_{(B6)} \underbrace{1}_{(B7)},$$

with $(\alpha_1, \alpha_2, \alpha_3) = (7, 4, 3)$ and $(\beta_1, \beta_2, \beta_3, \beta_4) = (2, 2, 4, 6)$. Since (20) holds, $\mathcal{I}(\sigma) = \{(6, 11)\}$.

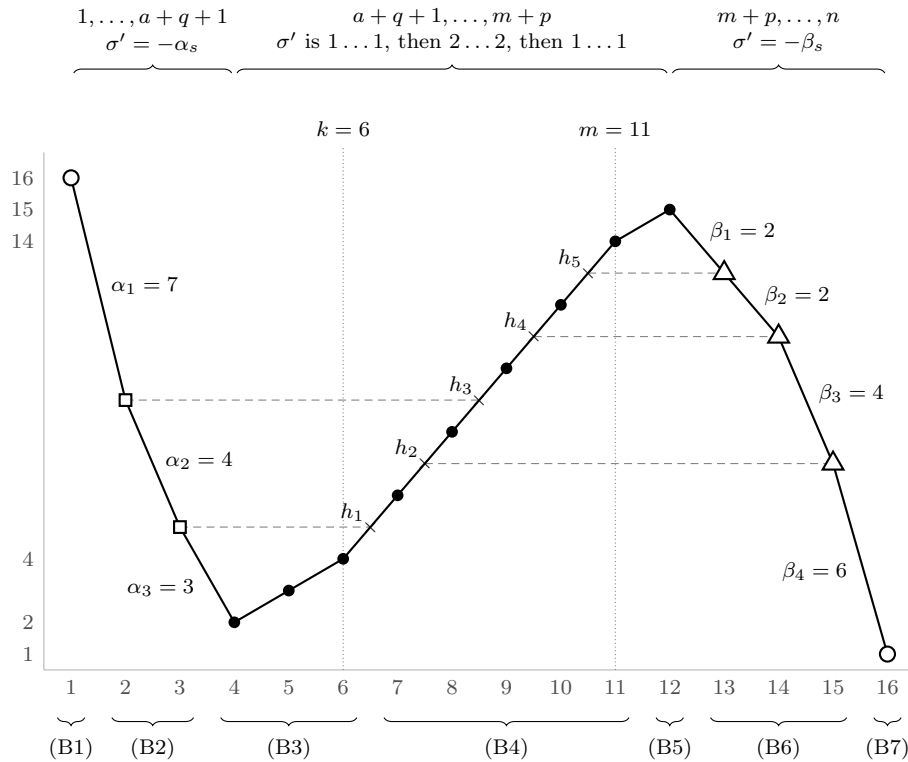


FIGURE 6. The permutation of Example 5.9, $\sigma = 16 \ 9 \ 5 \ 2 \ 3 \ 4 \ 6 \ 8 \ 10 \ 12 \ 14 \ 15 \ 13 \ 11 \ 7 \ 1$.

Proof. We first show that $\sigma \in \text{Sym}_n$. The entries of (B3) are the elements of $\{c+1, \dots, x\}$, and the entries of (B5) are the elements of $\{x+2\ell+1, \dots, x+2\ell+p\}$. The entries of (B4), (B2) and (B6) are the numbers $x+2i$ and $h_i = x+2i-1$ for $1 \leq i \leq \ell$, which together with x are the elements of $\{x, \dots, x+2\ell\}$. Finally, (B7) contributes 1 exactly when $c = 1$, that is, when $c+1 > 1$, and (B1) contributes n exactly when $a = 1$, that is, when $n > x+2\ell+p$. Hence every element of $[1, n]$ appears exactly once in σ .

Next we locate the blocks. Blocks (B1) and (B2) have $a+q$ entries, so (B3) occupies the positions $a+q+1, \dots, a+q+r+1 = k$, and (B4) occupies positions $k+1, \dots, k+\ell = m$. In

particular, $\sigma(k) = x$ and $\sigma(m) = x + 2\ell$. Block (B5) occupies the positions $m + 1, \dots, m + p$, block (B6) occupies the positions $m + p + 1, \dots, m + p + t$, and $n = m + p + t + c$. Since $t \geq 1$, we have $m \leq n - 1$. The formula $n = 2\ell + r + a + p + c + 1$ follows from $n = x + 2\ell + p + a$ and $x = c + r + 1$.

We now compute σ' and prove (19). We split the positions of σ into three ranges.

- Positions $1, \dots, a + q + 1$: here, the entries of σ are, in order, n (only if $a = 1$), $h_{i_1}, h_{i_2}, \dots, h_{i_q}$ (only if $q \geq 1$), and $c + 1$.

This list is strictly decreasing. Indeed, if $a = 1$ then $n = x + 2\ell + p + 1 > h_\ell$, so n is larger than every h_i and than $c + 1$. Moreover, $h_{i_1} > h_{i_2} > \dots > h_{i_q}$ because $i_1 > i_2 > \dots > i_q$. Finally, every h_i satisfies $h_i \geq h_1 = x + 1 > c + 1$. (If $q = a = 0$, this range is the single position 1 and there are no α_s .) Therefore, for $1 \leq s \leq q + a$,

$$\alpha_s = \sigma(s) - \sigma(s + 1) > 0 \quad \text{and} \quad \sigma'(s) = -\alpha_s.$$

- Positions $a + q + 1, \dots, m + p$. Here, the entries of σ are those of the blocks (B3), (B4) and (B5), that is, in order,

$$c + 1, c + 2, \dots, x, \quad x + 2, x + 4, \dots, x + 2\ell, \quad x + 2\ell + 1, x + 2\ell + 2, \dots, x + 2\ell + p.$$

Two consecutive entries of this list differ by 1 (the first r times), then by 2 (the next ℓ times), then by 1 (the last p times). Since $a + q + r + 1 = k$ and $k + \ell = m$, this means that

$$\sigma'(s) = \begin{cases} 1 & \text{for } a + q + 1 \leq s \leq k - 1, \\ 2 & \text{for } k \leq s \leq m - 1, \\ 1 & \text{for } m \leq s \leq m + p - 1. \end{cases}$$

- Positions $m + p, \dots, n$. Here, the entries of σ are, in order,

$$x + 2\ell + p, \quad h_{j_1}, h_{j_2}, \dots, h_{j_t}, \quad 1 \quad (\text{only if } c = 1).$$

This list is strictly decreasing: $x + 2\ell + p > h_\ell \geq h_{j_1}$, $h_{j_1} > h_{j_2} > \dots > h_{j_t}$ because $j_1 > j_2 > \dots > j_t$, and $h_{j_t} \geq h_1 = x + 1 > 1$. Therefore, for $1 \leq s \leq t + c$,

$$\beta_s = \sigma(m + p + s - 1) - \sigma(m + p + s) > 0 \quad \text{and} \quad \sigma'(m + p + s - 1) = -\beta_s.$$

Reading the three ranges in order, we obtain (19).

Finally, assume (20). We check the three conditions of Lemma 5.3. By (19), $\sigma'(j) = 2$ for $k \leq j \leq m - 1$, so that $d = \sigma(k + 1) - \sigma(k) = 2 > 0$ and (4) holds. On $[1, k - 1]$, σ' is $-\alpha_1 \leq -\alpha_2 \leq \dots \leq -\alpha_{q+a} < 0$, followed by r entries equal to 1, so (3) holds. On $[m, n - 1]$, σ' is p entries equal to 1 followed by $0 > -\beta_1 \geq -\beta_2 \geq \dots \geq -\beta_{t+c}$, so (5) holds. By Lemma 5.3, $\mathcal{I}(\sigma) = \{(k, m)\}$. $\square$

We can now state and prove the first main result of this section, which fully characterizes, for $m \geq 2k + 1$, when $\{\sigma \in \text{Sym}_n \mid \mathcal{I}(\sigma) = \{(k, m)\}\} \neq \emptyset$. The idea of the proof is as follows (see Figure 7): outside of case (5) below, the progression $\sigma(k), \dots, \sigma(m)$ has $d = 2$, so it skips $\ell = m - k$ values, and each of them must be used before or after the progression. Hence $n = 2\ell + N$, where N counts the entries that are neither in the progression after $\sigma(k)$ nor skipped by it, and the monotonicity of σ' before k and after m leaves space for only a few such entries.

Theorem 5.10. *Let $k \geq 2$, $m \geq 2k + 1$ and $n \geq m + 1$, and write $n = 2(m - k) + N$. Then $|\{\sigma \in \text{Sym}_n \mid \mathcal{I}(\sigma) = \{(k, m)\}\}| > 0$ if and only if one of the following holds:*

- (1) $1 \leq N \leq \min\{7, k + 2\}$;
- (2) $k - 1 \leq N \leq k + 2$;
- (3) $N = 8$ and $m \in \{3k - 10, 3k - 8\}$;

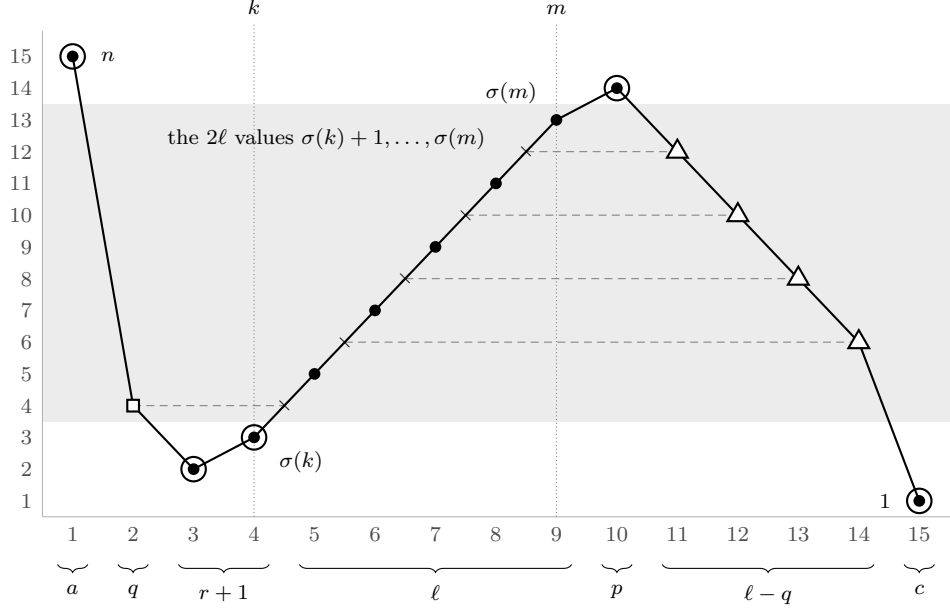


FIGURE 7. The idea of the proof of Theorem 5.10, for $\sigma = 15\ 4\ 2\ 3\ 5\ 7\ 9\ 11\ 13\ 14\ 12\ 10\ 8\ 6\ 1$, where $(k, m, n) = (4, 9, 15)$ and $\ell = 5$. The shaded band contains 2ℓ values: the ℓ entries $\sigma(k+1), \dots, \sigma(m)$ of the progression and the ℓ values that it skips. Each skipped value is used before the progression ($\square$) or after it ($\triangle$). The remaining $N = n - 2\ell = r + a + p + c + 1$ entries are circled. Here $q = 1$, $r = a = p = c = 1$ and $N = 5$.

(4) $N = 9$ and $m = 3k - 10$;

(5) $k = 2$ and $n = m + 1$.

Proof. Write $\ell = m - k$, so that $\ell \geq k + 1 \geq 3$.

We first prove that the conditions are necessary. Let $\sigma \in \text{Sym}_n$ with $\mathcal{I}(\sigma) = \{(k, m)\}$. By Lemma 5.2, we may assume $d = \sigma(k+1) - \sigma(k) > 0$, so that σ is as in Lemma 5.4. Since $m > 2k$, Lemma 5.6 gives $d \leq 2$. If $d = 1$, Lemma 5.5 gives (5). Assume from now on that $d = 2$, and let $h_i, A, B, q, t, a, c, r, p$ be as in Lemma 5.7. By (9) and (10),

$$N = n - 2\ell = k - q + p + c = r + a + p + c + 1. \quad (21)$$

In particular, $N \geq 1$. Moreover, (9) gives $q \leq k - 1$, hence $t = \ell - q \geq 2$, and Lemma 5.7(e) gives $p \leq 3$. We divide the proof in cases, depending on q .

Case $q = 0$: In this case $B = [1, \ell]$, so $j_1 - j_2 = 1$ and Lemma 5.7(e) gives $p \leq 1$. By (21), $N = k + p + c$, hence $k \leq N \leq k + 2$ and (2) holds.

Case $q = 1$: By (21), $N = k - 1 + p + c$, hence $k - 1 \leq N \leq k + 3$, and $N = k + 3$ only if $p = 3$ and $c = 1$. Assume by contradiction that $p = 3$ and $c = 1$. By Lemma 5.7(e), $B = \{\ell, \ell - 2, \dots, \ell - 2(t - 1)\}$ with $t = \ell - 1$. Since $\ell - 2(t - 1) \geq 1$, we obtain $\ell \leq 3$, hence $\ell = 3$, $k = 2$ and $B = \{3, 1\}$. By (9), $r = a = 0$. By (12), $\sigma(n - 2) = h_3$, $\sigma(n - 1) = h_1$ and $\sigma(n) = 1$, and by (13), $|\sigma'(n - 2)| = 4$ and $|\sigma'(n - 1)| = h_1 - 1 = 2$. This contradicts the fact that $|\sigma'|$ is non-decreasing on $[m^*, n - 1]$. Therefore $k - 1 \leq N \leq k + 2$ and (2) holds.

Case $q \geq 2$: By (21), $N = k - q + p + c \leq k - 2 + 3 + 1 = k + 2$. By Lemma 5.7(d), $r \leq 3$, so (21) also gives $N = r + a + p + c + 1 \leq 9$. If $N \leq 7$, then (1) holds. Assume now $N \geq 8$. Then $r + p \geq 5$, and since $r \leq 3$ and $p \leq 3$, we must have $r \geq 2$ and $p \geq 2$. By Lemma 5.7(d), $1 \in A$, $3 \in A$ and $2 \notin A$, that is, $2 \in B$. By Lemma 5.7(e), $B = \{\ell, \ell - 2, \dots, \ell - 2(t - 1)\}$. Since $2 \in B$, ℓ is even and $B = \{2, 4, \dots, \ell\}$, so $q = \ell/2$. By (9),

$$\ell = 2q = 2(k - 1 - r - a).$$

If $N = 9$, then $r = 3$, $a = 1$, $p = 3$ and $c = 1$, hence $\ell = 2k - 10$, that is, $m = 3k - 10$, and (4) holds. If $N = 8$, then (r, a, p, c) is one of $(3, 1, 3, 0)$, $(3, 1, 2, 1)$, $(3, 0, 3, 1)$, $(2, 1, 3, 1)$, hence $\ell \in \{2k - 10, 2k - 8\}$, that is, $m \in \{3k - 10, 3k - 8\}$, and (3) holds.

This proves that one of (1)–(5) holds.

We now prove that the conditions are sufficient. If (5) holds, then $\sigma = (m + 1) 2 3 \dots m 1 \in \text{Sym}_{m+1}$ has first derivative

$$(1 - m) \underbrace{1 \dots 1}_{m-2} (1 - m),$$

so that $\mathcal{I}(\sigma) = \{(2, m)\}$ by Lemma 5.3 (with $d = 1$).

For the remaining cases, we use Lemma 5.8, with three different choices of sets A and B . For each choice, we say which missing values go in the blocks (B2) and (B6), and we compute the numbers α_s and β_s of that lemma, to find out when (20) holds. Throughout, $x = c + r + 1$, $h_i = x + 2i - 1$ and $n = x + 2\ell + p + a$, so that

$$h_i - h_{i'} = 2(i - i'), \quad h_i - (c + 1) = 2i - 1 + r, \quad (x + 2\ell + p) - h_i = 2(\ell - i) + p + 1,$$

and moreover $n - h_i = 2(\ell - i) + p + 2$ if $a = 1$, and $h_i - 1 = 2i + r$ if $c = 1$.

First choice: $A = \{1, \dots, q\}$ and $B = \{q + 1, \dots, \ell\}$, with $0 \leq q \leq k - 1$.

The block (B2) is $h_q h_{q-1} \dots h_1$, and the block (B6) is $h_\ell h_{\ell-1} \dots h_{q+1}$. If $q \geq 1$, then

$$(\alpha_1, \dots, \alpha_{q+a}) = \begin{cases} (\underbrace{2, \dots, 2}_{q-1}, 1 + r) & \text{if } a = 0, \\ (2(\ell - q) + p + 2, \underbrace{2, \dots, 2}_{q-1}, 1 + r) & \text{if } a = 1. \end{cases}$$

If $q = 0$, there are no α_s when $a = 0$, and there is only $\alpha_1 = n - (c + 1)$ when $a = 1$. In all cases,

$$(\beta_1, \dots, \beta_{t+c}) = \begin{cases} (p + 1, \underbrace{2, \dots, 2}_{\ell-q-1}) & \text{if } c = 0, \\ (p + 1, \underbrace{2, \dots, 2}_{\ell-q-1}, 2q + 2 + r) & \text{if } c = 1. \end{cases}$$

Notice that $\ell - q - 1 \geq 1$, since $q \leq k - 1 \leq \ell - 2$. Hence the second part of (20) holds if and only if $p \leq 1$. The first part of (20) holds in each of the following situations: if $q = 0$, or if $q = 1$ (when $a = 1$ it requires $1 + r \leq 2(\ell - 1) + p + 2$, which holds since $r \leq k - 2 \leq \ell - 3$), and if $q \geq 2$ and $r \leq 1$. In summary, with this choice, (20) holds if

$$p \leq 1, \quad \text{and} \quad r \leq 1 \quad \text{in case } q \geq 2.$$

Second choice: $A = \{1, 3, 5, \dots, 2q - 1\}$ and $B = \{2, 4, \dots, 2q\} \cup \{2q + 1, \dots, \ell\}$, with $q \geq 2$ and $\ell \geq 2q$.

The block (B2) is $h_{2q-1} h_{2q-3} \dots h_3 h_1$, and the block (B6) is $h_\ell h_{\ell-1} \dots h_{2q+1} h_{2q} h_{2q-2} \dots h_4 h_2$ (Example 5.9 is of this type, with $q = 2$). We have

$$(\alpha_1, \dots, \alpha_{q+a}) = \begin{cases} (\underbrace{4, \dots, 4}_{q-1}, 1+r) & \text{if } a = 0, \\ (2(\ell - 2q) + p + 4, \underbrace{4, \dots, 4}_{q-1}, 1+r) & \text{if } a = 1, \end{cases}$$

and

$$(\beta_1, \dots, \beta_{t+c}) = \begin{cases} (p+1, \underbrace{2, \dots, 2}_{\ell-2q}, \underbrace{4, \dots, 4}_{q-1}) & \text{if } c = 0, \\ (p+1, \underbrace{2, \dots, 2}_{\ell-2q}, \underbrace{4, \dots, 4}_{q-1}, 4+r) & \text{if } c = 1. \end{cases}$$

The first part of (20) holds if and only if $1 + r \leq 4$. The second part holds if and only if $p + 1 \leq 2$ (when $\ell > 2q$) or $p + 1 \leq 4$ (when $\ell = 2q$). In summary, with this choice, (20) holds if

$$r \leq 3, \quad \text{and} \quad p \leq 1, \quad \text{or} \quad p \leq 3 \quad \text{and} \quad \ell = 2q.$$

Third choice: $B = \{j, j+2, j+4, \dots, \ell\}$ and $A = \{1, \dots, j-1\} \cup \{j+1, j+3, \dots, \ell-1\}$, where $j = 2q + 2 - \ell$ and $q + 2 \leq \ell \leq 2q$.

Notice that $j \geq 2$, that B has $\ell - q$ elements, and that A has q elements. The block (B2) is $h_{\ell-1} h_{\ell-3} \dots h_{j+1} h_{j-1} h_{j-2} \dots h_1$, and the block (B6) is $h_\ell h_{\ell-2} \dots h_{j+2} h_j$. We have

$$(\alpha_1, \dots, \alpha_{q+a}) = \begin{cases} (\underbrace{4, \dots, 4}_{\ell-q-1}, \underbrace{2, \dots, 2}_{j-2}, 1+r) & \text{if } a = 0, \\ (p+4, \underbrace{4, \dots, 4}_{\ell-q-1}, \underbrace{2, \dots, 2}_{j-2}, 1+r) & \text{if } a = 1, \end{cases}$$

and

$$(\beta_1, \dots, \beta_{t+c}) = \begin{cases} (p+1, \underbrace{4, \dots, 4}_{\ell-q-1}) & \text{if } c = 0, \\ (p+1, \underbrace{4, \dots, 4}_{\ell-q-1}, 2j+r) & \text{if } c = 1. \end{cases}$$

Notice that $\ell - q - 1 \geq 1$ and $2j + r \geq 4$. In summary, with this choice (20) holds if

$$r \leq 1 \quad \text{and} \quad p \leq 3.$$

In all three choices, Lemma 5.8 gives a permutation $\sigma \in \text{Sym}_n$ with $\mathcal{I}(\sigma) = \{(k, m)\}$ and $n = 2\ell + N$, where $N = r + a + p + c + 1$ and $q = k - 1 - r - a$. We now go through the values of N in (1)-(4).

- If $N = k - 1$, we use the first choice with $q = 1$, $a = 0$, $r = k - 2$ and $p = c = 0$.
- If $N \in \{k, k+1, k+2\}$, we use the first choice with $q = 0$, $a = 0$, $r = k - 1$, and $p, c \in \{0, 1\}$ with $p + c = N - k$.
- If $1 \leq N \leq \min\{5, k+2\}$, we use the first choice with $r, a, p, c \in \{0, 1\}$ and $r + a + p + c = N - 1$, where we take $r = a = 0$ if $N \leq 3$, and $r = 1$, $a = 0$ if $N = 4$. Then $q = k - 1 - r - a \geq 0$ (notice that if $N = 5$, then $k \geq 3$). For example, if $N = 1$, then $q = k - 1$ and $r = a = p =$

$c = 0$, and Lemma 5.8 gives $\sigma = (2k-2)(2k-4)\dots 2135\dots(2\ell+1)(2\ell)(2\ell-2)\dots(2k) \in \text{Sym}_{2\ell+1}$, with

$$\sigma' = \underbrace{(-2)\dots(-2)}_{k-2}(-1)\underbrace{2\dots 2}_{\ell}(-1)\underbrace{(-2)\dots(-2)}_{m-2k}.$$

- Let $N \in \{6, 7\}$. If $N \geq k-1$, we are in one of the first two items, so assume $N \leq k-2$, hence $k \geq 8$. If $\ell \geq 2k-10$, we use the second choice with $r = 3$, $a = 1$, $p = 1$, $c = N-6$ and $q = k-5$. If $\ell \leq 2k-11$, we use the third choice with $r = 1$, $a = 1$, $p = 3$, $c = N-6$ and $q = k-3$ (notice that $q+2 \leq \ell \leq 2q$ since $\ell \geq k+1$).
- If $N \in \{8, 9\}$ and $m = 3k-10$, then $\ell = 2k-10$, and we use the second choice with $q = k-5 = \ell/2$, $r = 3$, $a = 1$, $p = 3$ and $c = N-8$.
- If $N = 8$ and $m = 3k-8$, then $\ell = 2k-8$, and we use the second choice with $q = k-4 = \ell/2$, $r = 3$, $a = 0$, $p = 3$ and $c = 1$.

In the last two items, $q = \ell/2 \geq 2$ since $\ell \geq 3$. □

Corollary 5.11. *Let $k \geq 2$ and $m \geq 2k+1$.*

- If $\sigma \in \text{Sym}_n$ and $\mathcal{I}(\sigma) = \{(k, m)\}$, then $2m-2k+1 \leq n \leq 2m-k+2$, or $k = 2$ and $n = m+1$. In particular, $|\{\sigma \in \text{Sym}_n \mid \mathcal{I}(\sigma) = \{(k, m)\}\}| = 0$ for all $n \geq 2m-k+3$, and this is sharp.*
- There exists $\sigma \in \text{Sym}_n$ with $\mathcal{I}(\sigma) = \{(k, m)\}$ for every $2m-2k+1 \leq n \leq 2m-k+2$ if and only if $k \leq 9$ or $(k, m) \in \{(10, 22), (11, 23)\}$.*
- If $k \geq 10$ and $n = 2m-2k+N$ with $8 \leq N \leq k-2$, there exists no $\sigma \in \text{Sym}_n$ with $\mathcal{I}(\sigma) = \{(k, m)\}$, except when $N = 8$ and $m \in \{3k-10, 3k-8\}$, or $N = 9$ and $m = 3k-10$. For instance, there is no $\sigma \in \text{Sym}_{30}$ with $\mathcal{I}(\sigma) = \{(10, 21)\}$, although such permutations exist in Sym_{29} and in Sym_{31} .*

Proof. Writing $n = 2(m-k) + N$, the inequalities $2m-2k+1 \leq n \leq 2m-k+2$ are equivalent to $1 \leq N \leq k+2$. Hence (a) and (c) follow from Theorem 5.10. Notice that (a) is sharp since $N = k+2$ satisfies Theorem 5.10(2).

For (b), conditions (1) and (2) from Theorem 5.10 cover all $1 \leq N \leq k+2$ exactly when $k \leq 9$. If $k \geq 10$, the values $8 \leq N \leq k-2$ must satisfy (3) or (4) from Theorem 5.10. If $k \geq 12$, this fails for $N = 10$. If $k = 11$, $N = 9$ forces $m = 3k-10 = 23$, and then $N = 8$ satisfies (3) from Theorem 5.10. If $k = 10$, $N = 8$ forces $m \in \{20, 22\}$, and $m \geq 2k+1$ gives $m = 22$. Finally, $(k, m, n) = (10, 21, 30)$ has $N = 8$ and $m \notin \{20, 22\}$, while $n = 29$ and $n = 31$ correspond to $N = 7$ and $N = 9 = k-1$. □

Example 5.12. Consider the set $\{(3, 7)\}$. By Corollary 5.11, there exists $\sigma \in \text{Sym}_n$ with $\mathcal{I}(\sigma) = \{(3, 7)\}$ if and only if $9 \leq n \leq 13$.

We now turn to the case $k < m \leq 2k$. In Theorem 5.13, we construct permutations with the single inflection (k, m) . The construction itself depends on the relationship between k, m and n .

In (a), m is close to $2k$, so the number $\ell = m-k$ of values skipped by a progression with $d = 2$ is close to $k-1$, the number of positions before k . We fill these positions with skipped values in decreasing order, together with at most two other entries, and after the progression, σ simply increases by 1 for as long as needed. This is why every $n \geq m+1$ works, see Figures 8-11.

In (b), we use a progression with $d = 3$, which skips two values between any two consecutive entries. We use one of them before the progression and the other one after it (blocks P and U), so that σ decreases by 3 at the beginning and at the end. The values that are smaller than $\sigma(k)$ or

larger than $\sigma(m)$ are placed next to the progression (blocks Q and S), in an order that goes down by 2 and then up by 2 (respectively, up by 2 and then down by 2), which keeps σ' monotone. Since S can have any number of entries, this works for every $n \geq 2m - k$, see Figure 12.

In (c), we again use $d = 2$, but now only one skipped value is used before the progression, and all the others are used after it.

Theorem 5.13. *Let $2 \leq k < m \leq 2k$ and write $\ell = m - k$.*

- (a) *If $m \geq 2k - 3$ and $n \geq m + 1$, there exists $\sigma \in \text{Sym}_n$ with $\mathcal{I}(\sigma) = \{(k, m)\}$.*
- (b) *If $m \leq 2k - 1$ and $n \geq 2m - k$, there exists $\sigma \in \text{Sym}_n$ with $\mathcal{I}(\sigma) = \{(k, m)\}$.*
- (c) *If $m \geq k + 2$ and $n = 2m - k - 1$, there exists $\sigma \in \text{Sym}_n$ with $\mathcal{I}(\sigma) = \{(k, m)\}$.*

Proof. (a) We break into four cases depending on the value of m . In all of them, the arithmetic progression $\sigma(k), \dots, \sigma(m)$ has $d = 2$, the first derivative is non-decreasing and at most 1 on $[1, k - 1]$, and non-increasing and at most 1 on $[m, n - 1]$. Hence (3)–(5) hold with $d = 2$, and Lemma 5.3 gives $\mathcal{I}(\sigma) = \{(k, m)\}$.

When $m = 2k$, let $\sigma = (2k - 2)(2k - 4) \dots 2135 \dots (2k + 1)(2k + 2) \dots n(2k)$ (see Figure 8), where $\sigma(k) = 1$, $\sigma(m) = 2k + 1$, and

$$\sigma' = \underbrace{(-2) \dots (-2)}_{k-2} (-1) \underbrace{2 \dots 2}_k \underbrace{1 \dots 1}_{n-2k-1} (2k - n).$$

Notice that $2k - n \leq -1$ since $n \geq m + 1 = 2k + 1$.

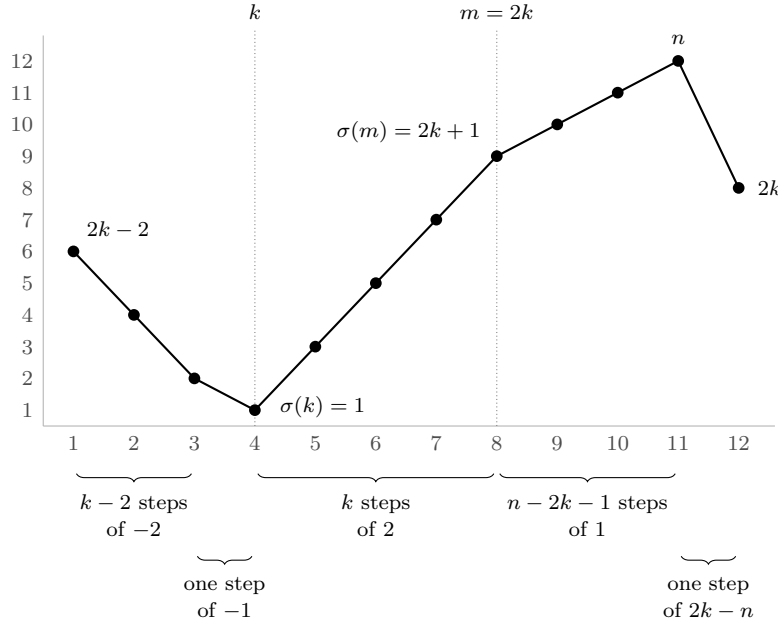


FIGURE 8. $\sigma = (2k - 2)(2k - 4) \dots 2135 \dots (2k + 1)(2k + 2) \dots n(2k)$ with $k = 4$ and $n = 12$, that is, $\sigma = 642135791011128$.

When $m = 2k - 1$, let $\sigma = (2k - 2)(2k - 4) \dots 2135 \dots (2k - 1)(2k)(2k + 1) \dots n$, see Figure 9. Then $\sigma \in \text{Sym}_n$, with $\sigma(k) = 1$ and $\sigma(m) = 2k - 1$, and σ has first derivative

$$\underbrace{(-2) \dots (-2)}_{k-2} (-1) \underbrace{2 \dots 2}_{k-1} \underbrace{1 \dots 1}_{n-2k+1}.$$

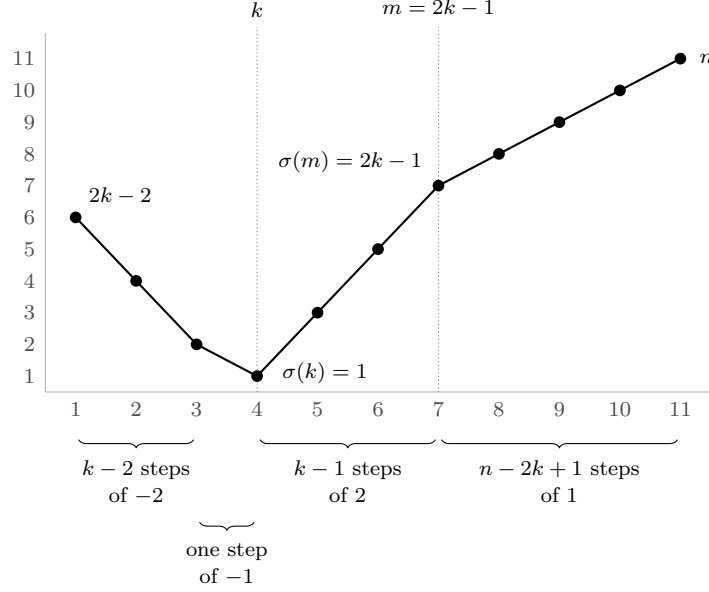


FIGURE 9. The permutation $\sigma = (2k-2)(2k-4)\dots 2135\dots(2k-1)(2k)(2k+1)\dots n$ with $k=4$ and $n=11$, that is, $\sigma = 6\ 4\ 2\ 1\ 3\ 5\ 7\ 8\ 9\ 10\ 11$.

When $m = 2k - 2$, let $\sigma = (2k-3)(2k-5)\dots 3124\dots(2k-2)(2k-1)(2k)\dots n$, see Figure 10. Then $\sigma \in \text{Sym}_n$, with $\sigma(k) = 2$ and $\sigma(m) = 2k - 2$, and σ has first derivative

$$\underbrace{(-2)\dots(-2)}_{k-2} \ 1 \ \underbrace{2\dots 2}_{k-2} \ \underbrace{1\dots 1}_{n-2k+2}.$$

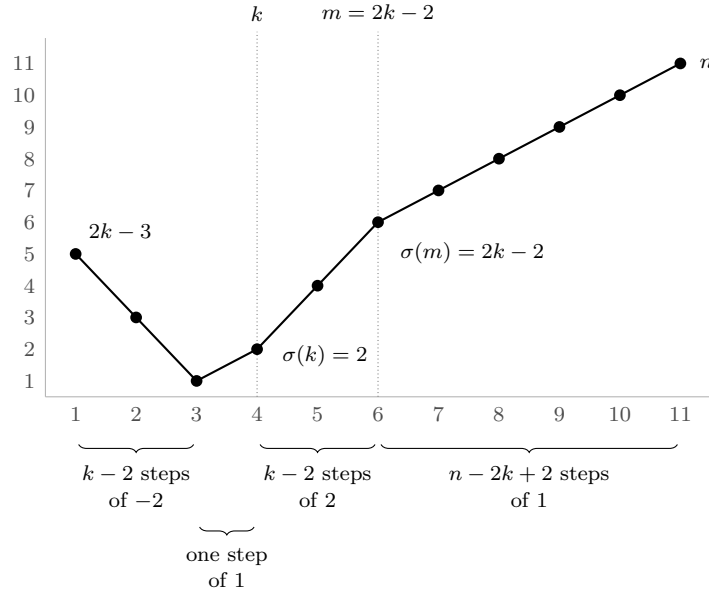


FIGURE 10. $\sigma = (2k-3)(2k-5)\dots 3124\dots(2k-2)(2k-1)(2k)\dots n$ with $k=4$ and $n=11$, that is, $\sigma = 5\ 3\ 1\ 2\ 4\ 6\ 7\ 8\ 9\ 10\ 11$.

When $m = 2k - 3$ (so that $k \geq 4$, since $m > k$), let $\sigma = n(2k - 5)(2k - 7) \dots 3124 \dots (2k - 4)(2k - 3)(2k - 2) \dots (n - 1)$, see Figure 11. Then $\sigma \in \text{Sym}_n$, with $\sigma(k) = 2$ and $\sigma(m) = 2k - 4$, and σ has first derivative

$$(2k - 5 - n) \underbrace{(-2) \dots (-2)}_{k-3} 1 \underbrace{2 \dots 2}_{k-3} \underbrace{1 \dots 1}_{n-2k+3},$$

where $2k - 5 - n \leq -3$ since $n \geq m + 1 = 2k - 2$.

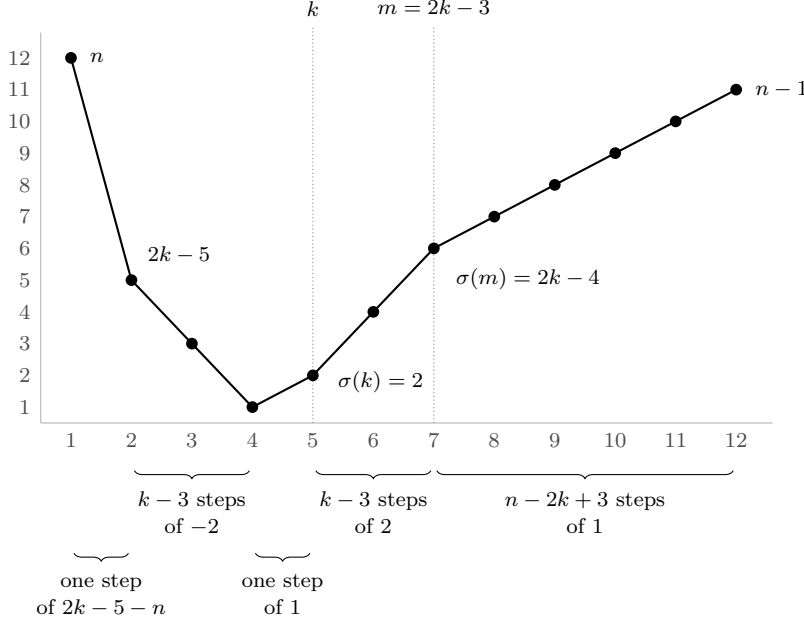


FIGURE 11. The permutation $\sigma = n(2k - 5)(2k - 7) \dots 3124 \dots (2k - 4)(2k - 3)(2k - 2) \dots (n - 1)$ with $k = 5$ and $n = 12$, that is, $\sigma = 125312467891011$.

(b) Let $b = 2k - m$ and $T = 2m - k$. Since $m \leq 2k - 1$ and $n \geq 2m - k$, we have $b \geq 1$ and $n \geq T$. Notice also that $T = b + 3\ell$. We define σ by listing, one after the other, the entries of five blocks,

$$\sigma = PQR SU,$$

which are defined as follows.

- $P = (T - 2)(T - 5)(T - 8) \dots (b + 4)(b + 1)$. It has ℓ entries, and it decreases by 3.
- Q contains all the elements of $\{1, \dots, b - 1\}$, listed in two halves: first $(b - 1)(b - 3)(b - 5) \dots$, decreasing by 2 until it reaches 1 or 2, then the remaining elements, increasing by 2 until they reach $b - 2$. It has $b - 1$ entries (so Q is empty if $b = 1$, and $Q = 1$ if $b = 2$).
- $R = b(b + 3)(b + 6) \dots (b + 3\ell)$. It has $\ell + 1$ entries, it increases by 3, and its last entry is T .
- S contains all the elements of $\{T + 1, \dots, n\}$, listed in two halves: first $(T + 2)(T + 4)(T + 6) \dots$, increasing by 2 until it reaches $n - 1$ or n ; then the remaining elements, decreasing by 2 until they reach $T + 1$. It has $n - T$ entries (so S is empty if $n = T$, and $S = T + 1$ if $n = T + 1$).
- $U = (T - 1)(T - 4)(T - 7) \dots (b + 5)(b + 2)$. It has ℓ entries, and it decreases by 3.

For example, if $(k, m, n) = (7, 10, 16)$, then $\ell = 3$, $b = 4$, $T = 13$, and

$$\sigma = \underbrace{11 \ 8 \ 5}_P \underbrace{3 \ 1 \ 2}_Q \underbrace{4 \ 7 \ 10 \ 13}_R \underbrace{15 \ 16 \ 14}_S \underbrace{12 \ 9 \ 6}_U,$$

with $\sigma' = (-3)(-3)(-2)(-2)1233321(-2)(-2)(-3)(-3)$ (see Figure 12).

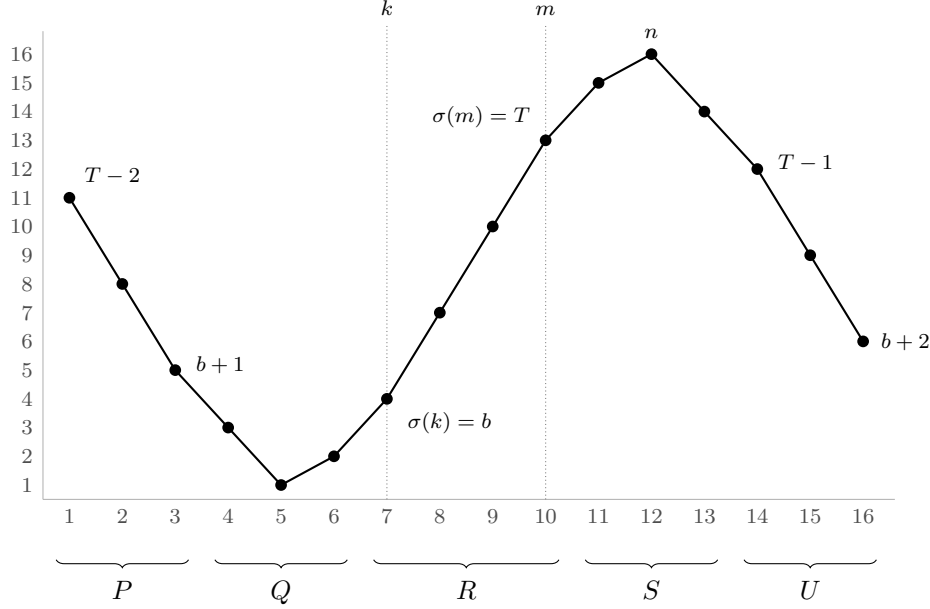


FIGURE 12. The permutation $\sigma = PQRSU$ with $(k, m, n) = (7, 10, 16)$, so that $\sigma = 11 \ 8 \ 5 \ 3 \ 1 \ 2 \ 4 \ 7 \ 10 \ 13 \ 15 \ 16 \ 14 \ 12 \ 9 \ 6$.

We first show that $\sigma \in \text{Sym}_n$, and that R occupies the positions $k, \dots, m$. The entries of Q are the elements of $\{1, \dots, b-1\}$, and the entries of S are the elements of $\{T+1, \dots, n\}$. Among the elements of $\{b, \dots, T\}$, those congruent to $b, b+1$ and $b+2$ modulo 3 are exactly the entries of R, P and U , respectively. Hence, every element of $[1, n]$ appears exactly once in σ . Moreover, P and Q together have $\ell + (b-1) = k-1$ entries, so the $\ell+1$ entries of R occupy the positions $k, \dots, k+\ell = m$. In particular, $\sigma(k) = b$, $\sigma(m) = T$, and $\sigma'(j) = 3$ for $k \leq j \leq m-1$. Finally, $m \leq n-1$, since U is not empty.

We now compute $\sigma'(j) = \sigma(j+1) - \sigma(j)$ on $[1, k-1]$. If $\sigma(j)$ and $\sigma(j+1)$ are both in P , then $\sigma'(j) = -3$, if they are both in the first half of Q , then $\sigma'(j) = -2$, and if they are both in the second half of Q , then $\sigma'(j) = 2$. It remains to compute $\sigma'(j)$ when $\sigma(j)$ is the last entry of a block (or of the first half of Q) and $\sigma(j+1)$ is the first entry of the next one.

- If $b \geq 2$, the last entry of P is $b+1$ and the first entry of Q is $b-1$, so here $\sigma'(j) = -2$.
- If $b \geq 3$, the first half of Q ends at 1 and the second half starts at 2 (when b is even), or the first half ends at 2 and the second half starts at 1 (when b is odd). So here $\sigma'(j) = (-1)^b$.
- If $b \geq 3$, the last entry of Q is $b-2$ and the first entry of R is b , so here $\sigma'(j) = 2$. If $b = 2$, then $Q = 1$ and the first entry of R is 2, so here $\sigma'(j) = 1 = (-1)^b$. If $b = 1$, then Q is empty, the last entry of P is 2 and the first entry of R is 1, so here $\sigma'(j) = -1 = (-1)^b$.

Counting the entries of each kind, we conclude that, in all cases, σ' restricted to $[1, k-1]$ is

$$\underbrace{(-3) \dots (-3)}_{\ell-1} \underbrace{(-2) \dots (-2)}_{\lceil (b-1)/2 \rceil} (-1)^b \underbrace{2 \dots 2}_{\lfloor (b-1)/2 \rfloor}. \quad (22)$$

We compute σ' on $[m, n-1]$ in the same way. If $\sigma(j)$ and $\sigma(j+1)$ are both in the first half of S , then $\sigma'(j) = 2$, if they are both in the second half of S , then $\sigma'(j) = -2$, and if they are both in U , then $\sigma'(j) = -3$. For the remaining positions:

- If $n \geq T+2$, the last entry of R is T and the first entry of S is $T+2$, so here $\sigma'(j) = 2$.
- If $n \geq T+2$, the first half of S ends at $n-1$ and the second half starts at n (when $n-T$ is odd), or the first half ends at n and the second half starts at $n-1$ (when $n-T$ is even). So here $\sigma'(j) = (-1)^{n-T+1}$.
- If $n \geq T+2$, the last entry of S is $T+1$ and the first entry of U is $T-1$, so here $\sigma'(j) = -2$. If $n = T+1$, then $S = T+1$, so $\sigma'(m) = 1 = (-1)^{n-T+1}$ and $\sigma'(m+1) = -2$. If $n = T$, then S is empty, the last entry of R is T and the first entry of U is $T-1$, so $\sigma'(m) = -1 = (-1)^{n-T+1}$.

Therefore, in all cases, σ' restricted to $[m, n-1]$ is

$$\underbrace{2 \dots 2}_{\lfloor (n-T)/2 \rfloor} (-1)^{n-T+1} \underbrace{(-2) \dots (-2)}_{\lceil (n-T)/2 \rceil} \underbrace{(-3) \dots (-3)}_{\ell-1}. \quad (23)$$

Putting everything together,

$$\sigma' = \underbrace{(-3) \dots (-3)}_{\ell-1} \underbrace{(-2) \dots (-2)}_{\lceil (b-1)/2 \rceil} (\pm 1) \underbrace{2 \dots 2}_{\lfloor (b-1)/2 \rfloor} \underbrace{3 \dots 3}_{\ell} \underbrace{2 \dots 2}_{\lfloor (n-T)/2 \rfloor} (\pm 1) \underbrace{(-2) \dots (-2)}_{\lceil (n-T)/2 \rceil} \underbrace{(-3) \dots (-3)}_{\ell-1}.$$

By (22), σ' is non-decreasing on $[1, k-1]$ and $\sigma'(k-1) \leq 2 < 3$, so (3) holds with $d = 3$. By (23), σ' is non-increasing on $[m, n-1]$ and $\sigma'(m) \leq 2 < 3$, so (5) holds. Since $\sigma'(j) = 3$ for $k \leq j \leq m-1$, Lemma 5.3 gives $\mathcal{I}(\sigma) = \{(k, m)\}$.

(c) Since $m \geq k+2$, we have $\ell \geq 2$ and $n = 2m - k - 1 = k + 2\ell - 1 \geq m + 1$. Let

$$\sigma = k \ 1 \ 2 \dots (k-1)(k+1)(k+3) \dots (k+2\ell-1)(k+2\ell-2)(k+2\ell-4) \dots (k+2).$$

See Figure 13.

Then $\sigma \in \text{Sym}_n$, with $\sigma(k) = k-1$ and $\sigma(m) = k+2\ell-1 = n$, and

$$\sigma' = (1-k) \underbrace{1 \dots 1}_{k-2} \underbrace{2 \dots 2}_{\ell} (-1) \underbrace{(-2) \dots (-2)}_{\ell-2}.$$

In particular, $\sigma'(j) = 2$ for $k \leq j \leq m-1$, the first derivative is non-decreasing and at most 1 on $[1, k-1]$, and non-increasing and at most -1 on $[m, n-1]$. Hence (3)–(5) hold with $d = 2$, and Lemma 5.3 gives $\mathcal{I}(\sigma) = \{(k, m)\}$. $\square$

When $m = k+1$, Theorem 5.13(b) gives a witness in Sym_n for every $n \geq m+1$. In this case there is also a simpler one: $\sigma = 1 \ 2 \dots k \ n \ (n-1)(n-2) \dots (k+1) \in \text{Sym}_n$ has

$$\sigma' = \underbrace{1 \dots 1}_{k-1} (n-k) \underbrace{(-1) \dots (-1)}_{n-k-1},$$

so that $\mathcal{I}(\sigma) = \{(k, k+1)\}$ by Lemma 5.3 (with $d = n-k \geq 2$). In particular, when $m = k+1$ the progression can have any $d \geq 2$, while $d \leq 2$ when $m \geq 2k+1$ by Lemma 5.6.

We are ready to fully characterize when $\{\sigma \in \text{Sym}_n \mid \mathcal{I}(\sigma) = \{(k, m)\}\} \neq \emptyset$, in the case $k < m \leq 2k$:

Theorem 5.14. *Let $k \geq 2$, $k < m \leq 2k$ and $n \geq m+1$. Then $|\{\sigma \in \text{Sym}_n \mid \mathcal{I}(\sigma) = \{(k, m)\}\}| > 0$ if and only if*

$$m \geq 2k-3 \quad \text{or} \quad n \geq 2m-k-1.$$

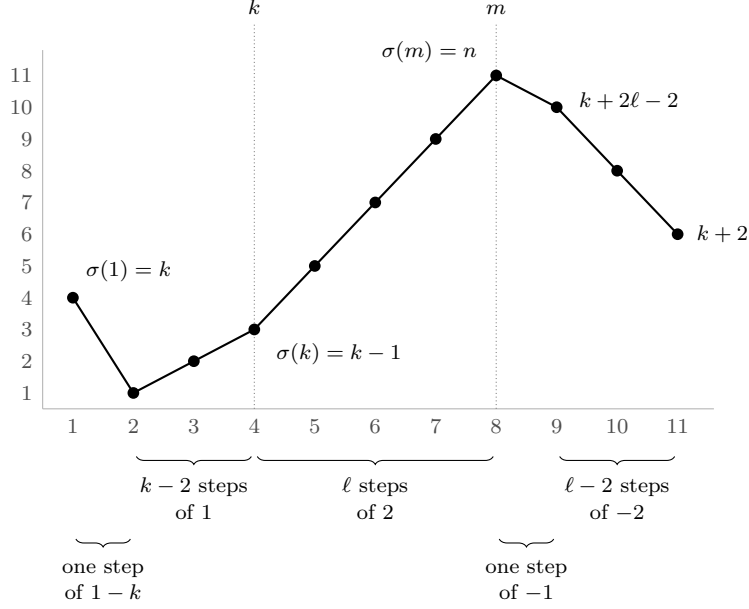


FIGURE 13. $\sigma = k 1 2 \dots (k-1)(k+1)(k+3) \dots (k+2\ell-1)(k+2\ell-2)(k+2\ell-4) \dots (k+2)$ with $k = 4$ and $m = 8$, so that $\ell = 4$ and $n = 11$, that is, $\sigma = 4 1 2 3 5 7 9 11 10 8 6$.

In other words, when $k < m \leq 2k$ there is a witness in Sym_n for every large n , and small values of n are excluded only when $m \leq 2k - 4$: in that case, the progression skips too few values to fill the positions before k , and we will show that at least $\ell - 1$ entries of σ must come after position m , that is, $n \geq m + \ell - 1 = 2m - k - 1$.

Proof. Write $\ell = m - k$.

Sufficiency: we need to find $\sigma \in \text{Sym}_n$ with $\mathcal{I}(\sigma) = \{(k, m)\}$ in each of the following cases.

- If $m \geq 2k - 3$, this is Theorem 5.13(a).
- If $m \leq 2k - 4$ and $n \geq 2m - k$, this is Theorem 5.13(b).
- If $m \leq 2k - 4$ and $n = 2m - k - 1$, then $n \geq m + 1$ gives $m \geq k + 2$, and this is Theorem 5.13(c).

Necessity: assume $m \leq 2k - 4$, and let $\sigma \in \text{Sym}_n$ with $\mathcal{I}(\sigma) = \{(k, m)\}$. We need to show that

$$n \geq 2m - k - 1.$$

Notice that $k < m \leq 2k - 4$ gives $k \geq 5$. By Lemma 5.2, we may assume $d = \sigma(k+1) - \sigma(k) > 0$, so that σ is as in Lemma 5.4. We divide the proof in cases, depending on d .

- Case $d = 1$: By Lemma 5.5, this would give $k = 2$, contradicting $k \geq 5$. So this case does not occur.
- Case $d \geq 3$: By Lemma 5.6, $n \geq 2m - k - 1$.
- Case $d = 2$: Let $h_i, A, B, q, t, a, c, r, p$ be as in Lemma 5.7. Since $m + \ell = 2m - k$, (10) gives

$$n = m + p + \ell - q + c = 2m - k + (p + c - q).$$

Hence $n \geq 2m - k - 1$ is equivalent to

$$q \leq p + c + 1. \quad (24)$$

If $q \leq 1$, then (24) holds, since $p \geq 0$ and $c \geq 0$. Assume from now on that $q \geq 2$, and assume by contradiction that (24) fails, that is,

$$q \geq p + c + 2. \quad (25)$$

Since $A \subseteq [1, \ell]$, we have $q \leq \ell = m - k \leq k - 4$. By (9), $r + a = k - 1 - q \geq 3$. Since $a \leq 1$, we conclude that $r \geq 2$. Now, since $q \geq 2$ and $r \geq 2$, Lemma 5.7(d) gives $i_q = 1$ and $i_{q-1} = 3$. These are the two smallest elements of A , so $1 \in A$, $3 \in A$ and $2 \notin A$. Since $2 \notin A$, we have $q \leq \ell - 1 \leq k - 5$, so (9) gives $r + a = k - 1 - q \geq 4$. On the other hand, $r \leq 3$ by Lemma 5.7(d), and $a \leq 1$. Hence $r = 3$, $a = 1$ and $q = k - 5 = \ell - 1$. Since A has $\ell - 1$ elements and $2 \notin A$, we conclude that $A = [1, \ell] \setminus \{2\}$.

Since $3 \in A$, we have $\ell \geq 3$. If $\ell \geq 4$, then $3, 4 \in A$ by what we have just shown, and Lemma 5.7(a) (applied with $i = 4$) gives $\{1, 2, 3, 4\} \subseteq A$, contradicting the fact that $2 \notin A$. Hence $\ell = 3$, $A = \{1, 3\}$ and $q = 2$. Finally, since $a = 1$ and $A = \{1, 3\}$, (11) gives

$$\sigma(1) = n, \quad \sigma(2) = h_3, \quad \sigma(3) = h_1,$$

and $k^* - 1 = q + a = 3$. By (13) (with $\ell = 3$ and $a = 1$),

$$|\sigma'(1)| = n - h_3 = p + 2 \quad \text{and} \quad |\sigma'(2)| = h_3 - h_1 = 4.$$

Since $|\sigma'|$ is non-increasing on $[1, k^* - 1] = [1, 3]$, we have $p + 2 \geq 4$, that is, $p \geq 2$. However, (25) and $q = 2$ give $p + c \leq 0$, that is, $p = 0$. This is a contradiction, so (24) holds.

In all cases, $n \geq 2m - k - 1$. □

We summarize Theorems 5.10 and 5.14 in the following corollary.

Corollary 5.15. *Let $2 \leq k < m$ and $I = \{(k, m)\}$. Then I is an admissible inflection set. Moreover, I is vanishing if and only if $m \geq 2k + 1$.*

Proof. If $m \geq 2k + 1$, Theorem 5.10(1) (with $N = 1$) gives a witness in Sym_n for $n = 2(m - k) + 1$, and Corollary 5.11(a) shows that there is no witness in Sym_n for $n \geq 2m - k + 3$. Hence I is admissible and vanishing. If $k < m \leq 2k$, Theorem 5.14 gives a witness in Sym_n for every $n \geq \max\{m + 1, 2m - k - 1\}$. Hence I is admissible and it is not vanishing. □

6. MAXIMALLY INFLECTED PERMUTATIONS

In this section, we study the size of $\mathcal{M}_n$, the set of maximally inflected permutations in Sym_n . Our last main result, Theorem 6.15, proves the explicit bound $|\mathcal{M}_n|/n! \leq (2/3)^{\lfloor n/3 \rfloor - 1}$. To define a maximally inflected permutation, we start with a simple lemma:

Lemma 6.1. *If $\sigma \in \text{Sym}_n$, then $|\mathcal{I}(\sigma)| \leq n - 3$.*

Proof. If $(i, j) \in \mathcal{I}(\sigma)$, then j is the smallest index larger than i with $\sigma''(j) \neq 0$, so j is determined by i . Hence $(i, j) \mapsto i$ is injective on $\mathcal{I}(\sigma)$, with image contained in $[2, n - 2]$, and $|\mathcal{I}(\sigma)| \leq n - 3$. □

Definition 6.2. For $n \geq 4$, if $\sigma \in \text{Sym}_n$ and $|\mathcal{I}(\sigma)| = n - 3$, we say σ is *maximally inflected*. We will write $\mathcal{M}_n = \{\sigma \in \text{Sym}_n : \sigma \text{ is maximally inflected}\}$.

Definition 6.3. A finite list $t_1, \dots, t_N$ is called *zigzag* if $(t_{j+1} - t_j)(t_{j+2} - t_{j+1}) < 0$ for $1 \leq j \leq N - 2$, that is, it is strictly alternately increasing and decreasing.

Theorem 6.4. *For $\sigma \in \text{Sym}_n$ the following are equivalent:*

- (a) $\sigma \in \mathcal{M}_n$.

- (b) $\mathcal{I}(\sigma) = \{(2, 3), (3, 4), \dots, (n-2, n-1)\}$.
- (c) $\sigma''(\ell) \neq 0$ for all ℓ and $\sigma''(\ell)\sigma''(\ell+1) < 0$ for $2 \leq \ell \leq n-2$.
- (d) The first derivative $D\sigma = (s_1, \dots, s_{n-1})$ is zigzag.

Proof. The equivalence between (a), (b) and (c) is immediate in view of the definition of inflection and Lemma 6.1. To prove that (c) is equivalent to (d), notice that $\sigma''(\ell) = s_\ell - s_{\ell-1}$, hence (c) is exactly (d). $\square$

Corollary 6.5. *Every alternating permutation is maximally inflected, so*

$$|\mathcal{M}_n| \geq 2E_n,$$

where E_n is the n -th Euler zigzag number.

Proof. If σ is alternating, then the signs of $s_1, \dots, s_{n-1}$ alternate and $D\sigma$ is zigzag. By Theorem 6.4(d), the result follows. $\square$

Using code, we exhaustively enumerated the maximally inflected permutations for small n :

n	4	5	6	7	8	9	10	11
$ \mathcal{M}_n $	14	44	190	904	5028	30884	214598	1613896

While a closed formula for $|\mathcal{M}_n|$ for general n is still an open question, one can easily obtain initial bounds:

Remark 6.6. We have

$$4 \left(\frac{2}{\pi} \right)^{n+1} n! \lesssim |\mathcal{M}_n| \lesssim e^{-1/2} n!.$$

In fact, one has $E_n \sim 2 \left(\frac{2}{\pi} \right)^{n+1} n!$ (see, for example, [Sta10]), hence Corollary 6.5 gives the lower bound. For the upper bound, let

$$A(n) = \{\sigma \in \text{Sym}_n : \sigma \text{ contains three consecutive terms in arithmetic progression}\}.$$

By the method of moments (see [AS16], and also the discussion in [Lav26]), $|A(n)| \sim (1 - e^{-1/2})n!$ as $n \rightarrow \infty$. Since $\mathcal{M}_n \subseteq \text{Sym}_n \setminus A(n)$, we must have $|\mathcal{M}_n| \lesssim n! - (1 - e^{-1/2})n!$, hence

$$4 \left(\frac{2}{\pi} \right)^{n+1} n! \lesssim |\mathcal{M}_n| \lesssim e^{-1/2} n!$$

In particular, from the lower bound and Stirling's formula, $\lim_{n \rightarrow \infty} |\mathcal{M}_n|^{1/n} = \infty$.

To obtain more precise bounds for $|\mathcal{M}_n|$ we first introduce the *descent word of a permutation* (see [Sta10], and [BHR03], where descent words and their run words are used to count permutations with a prescribed pattern of rises and falls).

Definition 6.7. Given $\sigma \in \text{Sym}_n$, define the *descent word* of σ to be $w(\sigma) = w_1 w_2 \dots w_{n-1} \in \{u, d\}^{n-1}$ given by

$$w_j = \begin{cases} u & \text{if } \sigma(j) < \sigma(j+1) \\ d, & \text{if } \sigma(j) > \sigma(j+1). \end{cases}$$

Equivalently, $w_j = u$ if and only if $s_j > 0$, where $D\sigma = (s_1, \dots, s_{n-1})$ and $s_j = \sigma(j+1) - \sigma(j)$.

Definition 6.8. A *run* of $w(\sigma)$ is a maximal block of consecutive letters, that is, a set of consecutive indices $R = \{\ell, \ell + 1, \dots, m\} \subseteq \{1, \dots, n - 1\}$ with $w_\ell = w_{\ell+1} = \dots = w_m$ and $w_{\ell-1} \neq w_\ell$ (or $\ell = 1$), and $w_{m+1} \neq w_m$ (or $m = n - 1$). We define the *length of the run* to be $|R| = m - \ell + 1$.

Notice that the runs partition $\{1, \dots, n - 1\}$, so we can list them in order as $R_1, \dots, R_q$. We say that a run R_j is *interior* if $2 \leq j \leq q - 1$.

Definition 6.9. Given $n \geq 1$, define

$$\mathcal{O}_n = \{\sigma \in \text{Sym}_n : \text{every interior run of } w(\sigma) \text{ has odd length}\}.$$

Let $O(n) = |\mathcal{O}_n|$ and $p_n = \frac{O(n)}{n!}$.

Theorem 6.10. Given $n \geq 1$, we have $\mathcal{M}_n \subseteq \mathcal{O}_n$.

Proof. Let $\sigma \in \mathcal{M}_n$, and let $R = \{i, \dots, j\}$ be an interior run of $w(\sigma)$ (so $2 \leq i \leq j \leq n - 2$, $w_{i-1} \neq w_i$ and $w_{j+1} \neq w_j$). Then $\sigma''(i)$ and $\sigma''(j + 1)$ must have opposite signs. Since σ is maximally inflected, the sign of σ'' must alternate on $\{i, i + 1, \dots, j, j + 1\}$, hence $\text{sign}(\sigma''(j + 1)) = (-1)^{j+1-i} \text{sign}(\sigma''(i))$. This implies $(-1)^{j+1-i} = -1$, that is, the length of R is odd. $\square$

We now follow [Bón22] and [Kit11] to define the pattern of a permutation:

Definition 6.11. Given a list $x = (r_1, \dots, r_n)$ of n distinct real numbers, its *pattern/standardisation* is the permutation $\text{st}(x) \in \text{Sym}_n$ defined by

$$\text{st}(x)(i) = |\{j \in [n] : r_j \leq r_i\}|.$$

Given $\sigma \in \text{Sym}_n$ and $L = \{p, p + 1, \dots, q\}$, we write

$$\pi_L(\sigma) = \text{st}(\sigma(p), \sigma(p + 1), \dots, \sigma(q)).$$

Example 6.12. To fix ideas, let $n = 5$, $\sigma = 35412$, and $L = \{3, 4, 5\}$. In this case, $\pi_L(\sigma) = \text{st}(4, 1, 2) = 312$.

Theorem 6.13. For every $k, n \geq 2$, we have

$$|\mathcal{M}_n| \leq (n!) p_k^{\lfloor n/k \rfloor}. \quad (26)$$

Additionally, $p_{k+\ell} \leq p_k p_\ell$, hence

$$\limsup_{n \rightarrow \infty} \left(\frac{|\mathcal{M}_n|}{n!} \right)^{1/n} \leq \lim_{n \rightarrow \infty} p_n^{1/n}. \quad (27)$$

Proof. Fix $k, n \geq 2$. Let $\sigma \in \mathcal{M}_n$ and $L = \{p, p + 1, \dots, p + k - 1\}$ be k consecutive natural numbers. For simplicity of notation, write $\pi_L(\sigma) = \pi$, that is, $\pi \in \text{Sym}_k$ and $\text{st}(\sigma(p), \sigma(p + 1), \dots, \sigma(p + k - 1)) = \pi$. Following Definition 6.7, write $w(\pi) = (w_1, \dots, w_{k-1})$. We have $w(\pi)_l = w(\sigma)_{p+l-1}$.

Let $R = \{i, \dots, j\}$ be an interior run of $w(\pi)$ (so $2 \leq i$ and $j + 1 \leq k - 1$). Then $w_{p+(i-1)-1}$ and $w_{p+(j+1)-1}$ are letters of $w(\sigma)$ which differ from w_{p+i-1} and w_{p+j-1} . Therefore $\tilde{R} = \{p + i - 1, \dots, p + j - 1\}$ is an interior run of $w(\sigma)$. By Theorem 6.10, $|\tilde{R}| = |R|$ must be odd. We conclude that if $\sigma \in \mathcal{M}_n$, then $\pi_L(\sigma) \in \mathcal{O}_k$ for every set L of k consecutive natural numbers.

We now divide $\{1, \dots, n\}$ into m consecutive blocks $L_1, \dots, L_m$ of size k , where $m = \lfloor n/k \rfloor$, together (potentially) with a last block T of size less than k . Sampling σ uniformly from Sym_n is equivalent to sampling the ordered partition of values $(S_1, \dots, S_m, S_T)$ (where $S_j = \sigma(L_j)$ for $1 \leq j \leq m$) uniformly among all such partitions, and independently, choosing a uniform ordering

of each S_i and S_T . Given $1 \leq i \leq m$, determining whether $\pi_{L_i}(\sigma) \in \mathcal{O}_k$ or not depends only on the ordering chosen for S_i . We have

$$\Pr[\pi_{L_i}(\sigma) \in \mathcal{O}_k \mid S_1, \dots, S_m, S_T] = \frac{O(k)}{k!} = p_k.$$

Since the events are conditionally independent, we obtain (26):

$$\frac{|\mathcal{M}_n|}{n!} \leq \Pr[\cap_{j=1}^m \{\pi_{L_j}(\sigma) \in \mathcal{O}_k\}] = p_k^m.$$

To prove (27), we start by showing $p_{k+\ell} \leq p_k p_\ell$ for $k, \ell \geq 1$. Let $\zeta \in \mathcal{O}_{k+\ell}$. Divide $\{1, \dots, k+\ell\}$ into $L_1 = \{1, \dots, k\}$ and $L_2 = \{k+1, \dots, k+\ell\}$. The previous arguments show $\pi_{L_1}(\zeta) \in \mathcal{O}_k$ and $\pi_{L_2}(\zeta) \in \mathcal{O}_\ell$. Sampling ζ uniformly from $\text{Sym}_{k+\ell}$ gives

$$p_{k+\ell} = \Pr[\zeta \in \mathcal{O}_{k+\ell}] \leq \Pr[\pi_{L_1}(\zeta) \in \mathcal{O}_k \text{ and } \pi_{L_2}(\zeta) \in \mathcal{O}_\ell] = p_k p_\ell.$$

Applying Fekete's Lemma ([Ste97]) with $a_n = \log p_n$ allows us to conclude that $\lim_{n \rightarrow \infty} \frac{\log(p_n)}{n} = \inf \frac{\log(p_n)}{n}$, hence $\lim_{n \rightarrow \infty} p_n^{1/n}$ exists and

$$\lim_{n \rightarrow \infty} p_n^{\frac{1}{n}} = \inf p_n^{\frac{1}{n}}. \quad (28)$$

By Theorem 6.10, $\mathcal{M}_n \subseteq \mathcal{O}_n$, hence

$$\left(\frac{|\mathcal{M}_n|}{n!} \right)^{1/n} \leq p_n^{1/n}.$$

Together with (28), this gives (27). □

Lemma 6.14. *Let $S \subset \mathbb{R}$ be finite, with $|S| \geq 3$. Let (x, y, z) be a uniformly random ordered triple of distinct elements of S . Fix $s \in \{-1, 0, 1\}$. Then*

$$\Pr[\text{sign}(x - 2y + z) = s] \leq \frac{2}{3}.$$

Proof. Given $x, y, z \in S$, each of the six possible orderings between them is equally likely. There are two orderings where y is the smallest, and in that case, $x - 2y + z > 0$. There are two orderings where y is the largest, in which case $x - 2y + z < 0$. For the remaining two orderings, where y is the intermediate value, the sign of $x - 2y + z$ is the same. Therefore, for fixed $s \in \{-1, 1\}$ at most 4 orderings realize it, and only the two orderings where y is the intermediate value can give $x - 2y + z = 0$. Hence $\Pr[\text{sign}(x - 2y + z) = s] \leq 4/6 = 2/3$ for every $s \in \{-1, 0, 1\}$. □

We are now ready to prove the main result of this section:

Theorem 6.15. *Given $n \geq 3$,*

$$\frac{|\mathcal{M}_n|}{n!} \leq \left(\frac{2}{3} \right)^{\lfloor n/3 \rfloor - 1}. \quad (29)$$

In particular,

$$\lim_{n \rightarrow \infty} \frac{|\mathcal{M}_n|}{n!} = 0$$

and

$$\limsup_{n \rightarrow \infty} \left(\frac{|\mathcal{M}_n|}{n!} \right)^{1/n} \leq (2/3)^{1/3}.$$

Proof. Fix $n \geq 3$. By Theorem 6.4, $\sigma \in \mathcal{M}_n$ if and only if there exists $a \in \{-1, 1\}$ so that

$$\text{sign}(\sigma''(i)) = a(-1)^i \text{ for } 2 \leq i \leq n-1. \quad (30)$$

Call $k = \lfloor n/3 \rfloor$. If $k = 1$, then $n \in \{3, 4, 5\}$ and it is trivially true that $|\mathcal{M}_n| \leq n!$, so we can assume $k \geq 2$.

Given $1 \leq j \leq k$, let $L_j = \{3j-2, 3j-1, 3j\}$. Notice that the blocks $L_1, \dots, L_k$ are disjoint. Let $\sigma \in \mathcal{M}_n$ and $Q_j = \sigma''(3j-1)$. Condition (30) gives $\text{sign}(Q_j) = a(-1)^{3j-1} = a(-1)^{j-1} = (-1)^{j-1} \text{sign}(Q_1)$ for $1 \leq j \leq k$.

Calling $E_j = \{\sigma \in \text{Sym}_n : \text{sign}(Q_j) = (-1)^{j-1} \text{sign}(Q_1)\}$, we have proven that

$$\mathcal{M}_n \subseteq \bigcap_{j=2}^k E_j.$$

Let σ be a uniformly random permutation in Sym_n . Given $1 \leq j \leq k$, let $\mathcal{F}_j$ be the σ -algebra generated by the random vector $(\sigma(1), \dots, \sigma(3j))$ and $T_j = [n] \setminus \{\sigma(1), \dots, \sigma(3j-3)\}$, so $|T_j| \geq n - 3j + 3 \geq n - 3k + 3 \geq 3$. Notice that conditioned on $\mathcal{F}_{j-1}$ (for $2 \leq j \leq k$), the ordered triple $(\sigma(3j-2), \sigma(3j-1), \sigma(3j))$ is a uniformly random ordered triple of distinct elements of T_j . Moreover, $\text{sign}(Q_1)$ is $\mathcal{F}_1$ -measurable since $Q_1 = \sigma''(2) = \sigma(3) + \sigma(1) - 2\sigma(2)$. Since $\mathcal{F}_1 \subseteq \mathcal{F}_{j-1}$, $(-1)^{j-1} \text{sign}(Q_1)$ is $\mathcal{F}_{j-1}$ -measurable. Hence, given $2 \leq j \leq k$, Lemma 6.14 yields

$$\Pr[E_j \mid \mathcal{F}_{j-1}] \leq \frac{2}{3}. \quad (31)$$

Let $F_\ell = \bigcap_{j=2}^\ell E_j$. Using the law of iterated expectations,

$$\Pr[F_\ell] = \Pr[F_{\ell-1} \cap E_\ell] = \mathbb{E}[\mathbf{1}_{F_{\ell-1}} \mathbf{1}_{E_\ell}] = \mathbb{E}[\mathbb{E}[\mathbf{1}_{F_{\ell-1}} \mathbf{1}_{E_\ell} \mid \mathcal{F}_{\ell-1}]].$$

Now, if $\mathcal{G}$ is a σ -algebra and Y is $\mathcal{G}$ -measurable, then $\mathbb{E}[YX \mid \mathcal{G}] = Y \mathbb{E}[X \mid \mathcal{G}]$. We apply this with $Y = \mathbf{1}_{F_{\ell-1}}$ (which is $\mathcal{F}_{\ell-1}$ -measurable since $F_{\ell-1} = \bigcap_{j=2}^{\ell-1} E_j$), and obtain

$$\mathbb{E}[\mathbb{E}[\mathbf{1}_{F_{\ell-1}} \mathbf{1}_{E_\ell} \mid \mathcal{F}_{\ell-1}]] = \mathbb{E}[\mathbf{1}_{F_{\ell-1}} \mathbb{E}[\mathbf{1}_{E_\ell} \mid \mathcal{F}_{\ell-1}]] = \mathbb{E}[\mathbf{1}_{F_{\ell-1}} \Pr[E_\ell \mid \mathcal{F}_{\ell-1}]].$$

By (31), $\Pr[E_\ell \mid \mathcal{F}_{\ell-1}] \leq \frac{2}{3}$, hence by the monotonicity of expectation,

$$\mathbb{E}[\mathbf{1}_{F_{\ell-1}} \Pr[E_\ell \mid \mathcal{F}_{\ell-1}]] \leq \mathbb{E}[\mathbf{1}_{F_{\ell-1}} \cdot \frac{2}{3}] = \frac{2}{3} \mathbb{E}[\mathbf{1}_{F_{\ell-1}}] = \frac{2}{3} \Pr[F_{\ell-1}].$$

Putting everything together yields $\Pr[F_\ell] \leq \frac{2}{3} \Pr[F_{\ell-1}]$. Since

$$\Pr[F_2] = \Pr[E_2] = \mathbb{E}[\mathbf{1}_{E_2}] = \mathbb{E}[\mathbb{E}[\mathbf{1}_{E_2} \mid \mathcal{F}_1]] = \mathbb{E}[\Pr[E_2 \mid \mathcal{F}_1]] \leq \mathbb{E}\left[\frac{2}{3}\right] = \frac{2}{3},$$

we conclude $\Pr[F_\ell] \leq \left(\frac{2}{3}\right)^{\ell-1}$. Since $\mathcal{M}_n \subseteq \bigcap_{j=2}^k E_j$, we obtain (29):

$$\frac{|\mathcal{M}_n|}{n!} \leq \left(\frac{2}{3}\right)^{k-1}.$$

Finally, $k-1 \geq n/3-2$, hence $(2/3)^{k-1} \leq (9/4)(2/3)^{n/3}$, from which we conclude that

$$\limsup_{n \rightarrow \infty} \left(\frac{|\mathcal{M}_n|}{n!}\right)^{1/n} \leq \left(\frac{2}{3}\right)^{1/3}.$$

□

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